Statistics and learning

Tests

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- Examples of applications: decide if a new drug can be put on market after adequate clinical trials, decide if items comply with predefined standards, which genes are significantly differentially expressed in pathological cells ....

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- Examples of applications: decide if a new drug can be put on market after adequate clinical trials, decide if items comply with predefined standards, which genes are significantly differentially expressed in pathological cells ....
- Typically, sources to build hypothesis stem from quality need, values from a previous experiment, a theory that need experimental confirmation or an assumption based on observations.

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- some particular cases: one-sample, two-sample, paired tests; Z-tests, t-tests, χ^2-tests, F-tests ...

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Example 1: cheater detection

To introduce randomness, you are asked to throw a coin 200 times and write down the results. Why would I be suspicious about students that do not exhibit at least one HHHHHH or TTTTTT pattern ? Would I be (totally ?) fair if I was to blame (all of) them ?

Motivation 2

Example 2: rain makers

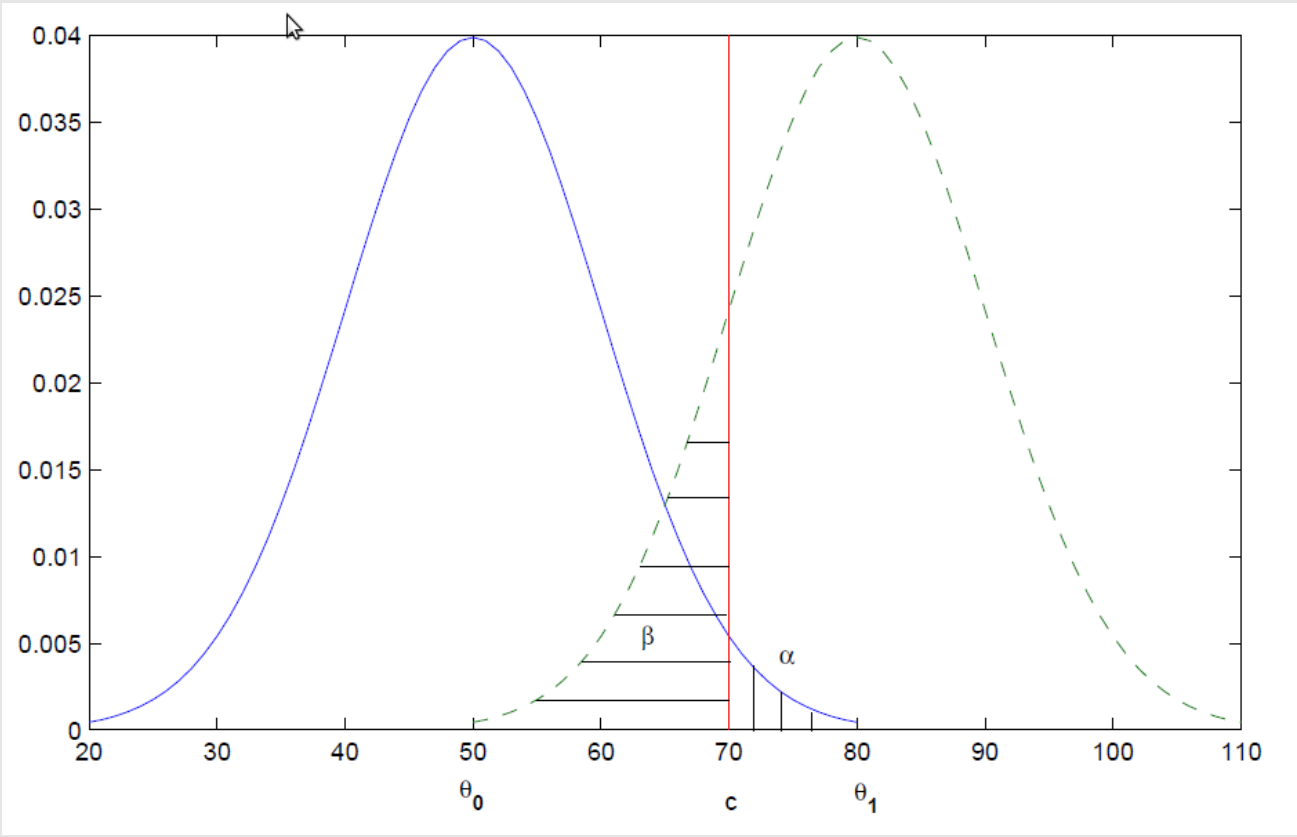
In a given area of agricultural interest, it usually rains $600mm$ a year. Suspicious scientists claim that they can locally increase rainfall, when spreading a revolutionary chemical (iodised silver) on clouds. Tests over the 1995-2002 period gave te following results:

Year	1995	1996	1997	1998	1999	2000	2001	2002
Rainfall (mm/year)	606	592	639	598	614	607	616	586

Does this sound correct to you ? Quantify the answer.
Bonus: what would have changed if you wanted to test if the increase was of say 30 mm ?

Motivation

Rain makers et possible errors



Hypothesis testing

(H0) $\theta = \theta_0$ and (H1) $\theta = \theta_1$

Tests

Possible situations

Real world / Decision made	(H0)	(H1)
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Apply that to "innoncent until proven guilty" and interpret the different situations. How do you want to control α and β ?
What about introducing a new drug on the market ??

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7. Strategic conclusion: how do YOU decide thanks to the light shed by statistical result ?

Methodology into details

- **Hypothesis**:= any subset of the family of all considered probability distributions $\mathcal{P}$. In practice, hypotheses are often on unknown parameters of distributions $\rightarrow$ parametric hypotheses, defined by equalities or inequalities: (H0) $\theta_0 \in \Theta_0$ and (H1) $\theta_0 \in \Theta_1$. In turn, they can be simple is only one value for the parameters is tested or muliple composite.

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- Choose a **test statistic** T_n:=a random variable which only depends on (Θ_0, Θ_1) and on obervations of the (X_i)'s. Interesting if the distribution is known given (H0) is true. Note that it is an estimator...depending on (H0) and (H1).

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- How to choose a good test statistic ? Remember the typology of confidence intervals ? And explore R help ?!

Methodology into details (cont'd)

- Determine the **rejection region** R. Usually of the form $(r; \infty)$, $(-\infty; r)$ or $(-\infty; r) \cup (r'; \infty)$. To decide, examine how the test statistic behaves under (H1).

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- **p-value**:=maximal value of α so that the test would accept the observed statistic to be drawn under (H0) $\approx$ credibility index on (H0). Alternative definition: probability to obtain a test statistic value at least as contradictory to (H0) as the observed value assuming (H0) is true if we repeated the experiment.

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- hence (H0) is chosen according to a firmly established theory (you don't want to make a fool of yourself), because caution is needed or...for subjective reasons (consumer choice is not that of manufacturers !)

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Historical notes: statistics were of great help in modern medicine.

Tests you need to know

and we shall see during the next session

- parametric tests (observations drawn from $\mathcal{N}$ or large samples so that C.L.Th. applies)
  - one sample: comparing the empirical mean to a theoretical value $\rightarrow$ Z-test or t-test
  - two independent samples: t-test, F-test
  - paired samples: paired t-test
  - several samples: ANOVA, not today !
- adequation tests: χ^2-test. Normality check: Kolmogorov or Shapiro-Wilks.
- non-parametric tests (when small samples or non Gaussian distributions)
  - comparing 2 medians from independent samples: Mann-Whitney test.
  - two paired samples: Wilcoxon test on differences.
  - several samples: Kruskal-Wallis

Exercises

Poisson arrival at motorway tolls

For two hours, at a motorway toll, we write down the number of cars arriving during each 2 minute intervals. We obtain:

#(cars)	0	1	2	3	4	5	6	7	8	9	10	11
#(intervals)	4	9	24	25	22	18	6	5	3	2	1	1

Test at a significance level of 0.1 the fit to a Poisson distribution with a parameter to be determined.

Evolution of purchasing power

In 2004, the total amount spent on products which are not essentials (*e.g.* travels, shows . . . as opposed to food, hoosing . . .) was 632 euros per month per household accoring to the INSEE during a partial survey over millions of households. In 2008, from a sample of 2000 interviewed by telephone, 1837 answers were obtained and the declared mean value was 598 euros (with sd 254 euros). If you assume a 2% inflation per year, would you say that the amount spent on non-essentials has significantly decreased ?

Finished

Next time: more tests and analysis of variance (ANOVA)