Statistics and learning

An introduction: from data to modelling

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Statistical approach

A quick, partial and not very comprehensive overview

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- Grail: linking data to mathematical modelling, objectivelly quantify and interpret conclusions and...awareness of limitations: **statistics helps but won't make decision for you** !

Inspiring work / our bibliography

Trevor Hastie, Robert Tibshirani and Jérome Friedman.
Elements of statistical learning.
Springer, 2nd edition, 2009.

Aurélien Garivier
Statistiques avancées.
Cours Centrale 2011, 2011.

Stéphan Clémençon.
Apprentissage statistique.
Cours TELECOM ParisTech, 2011-2012.

Sylvain Arlot, Francis Bach, Olivier Catoni, Gilles Stolz and Guillaume Obozinski
Apprentissage.
Cours ENS, 2012.

Nicolas Chopin, Dinah Rosenberg and Gilles Stolz
Eléments de statistique pour citoyens d'aujourd'hui et managers de demain.
Cours L3 HEC, 2012–2013.

Alain Baccini, Philippe Besse, Stéphane Canu, Sébastien Déjean, Béatrice Laurent, Clément Marteau, Pascal Martin and Hélène Milhem
Wikistat, le cours dont vous êtes le héros.
`http://wikistat.fr/`, 2012.

And many others we just forgot to mention.

From data to modelling

and back

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- empirical approach to gaining knowledge from an experiment repeated many times,

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Preference between two possible configurations

Consumer ID	1	2	3	4	5	6	...
Opinion	A	A	B	A	B	B	...

We can denote by x_i successive opinions taking (binary) values "A" $(= 0)$ or "B" $(= 1)$. Mathematician sees that as realisation of random variables denoted X_i.

Localising randomness

Randomness...

...arises from the choice of the questioned persons, NOT from in each actual answer.

Incidental reminder: Bernouilli distribution, with parameter $0 < p < 1$...

- laid question: is $p_0 > 1/2$ or $< 1/2$? This is a **test**.

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  2. quantitatively answer the question (generalising sample to full population conclusions)

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a mandatory step

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- remember the Jean Tibéri vs. Lyne Cohen-Solal (+ Ph. Meyer) council election in Paris in 2008 between 20.45 and 21.15 ? At 20.45, $(463; 409; 106)$ but after counting the votes : $(11,044; 11,269; 2,730)$.

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- Construction of **confidence intervals** to answer the question.

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- (almost never use skewness and kurtosis)

Two important probabilistic tools in statistics

Law of large numbers

Theorem

Let $X_1 \ldots X_n$ be iid random variables with mean μ. Then the empirical mean converges in probability towards μ, i.e.:

$$\bar{X_n} := \frac{1}{n}(X_1 + \ldots + X_n) \longrightarrow \mu.$$

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$$\bar{X_n} := \frac{1}{n}(X_1 + \ldots + X_n) \longrightarrow \mu.$$

In other term, for all $\epsilon > 0$, $P\left(\mid \bar{X_n} - \mu \mid > \epsilon\right) \to 0$

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Central limit theorem

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Let $X_1 \dots X_n$ be iid random variables which admit an order 2 moment. Denote by μ and σ the corresponding mean and standard deviation, then:

$$\frac{\sqrt{n}}{\sigma}(\bar{X}_n - \mu) \longrightarrow \mathcal{N}(0,1).$$

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In the case of distribution with density functions, this means that

$$P\left(\frac{\sqrt{n}}{\sigma}(\bar{X_n} - \mu) \leq x\right) := F_n(x) \longrightarrow P(Z \leq x) = \frac{\int_{-\infty}^{x} \mathrm{e}^{-z^2/2}\, dz}{\sqrt{2\pi}}.$$

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- Can we conclude ? Is this **estimate** enough ?

Let's play around the Central limit theorem...

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- is the conclusion similar if $n = 1,000$?

Note: 95% could have been replaced by 99%. How could this have affected the conclusion ? What about 100% ?

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- lessons from this: tests are not reductible to confidence intervals and...don't be fooled by an obscure choice of hypotheses !

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- ...and lots of R ;) !