Échantillonnage Monte-Carlo multi-niveaux pour la quantification d'incertitudes, l'analyse de sensibilité et l'assimilation de données

Application à la modélisation de l'écoulement des fleuves

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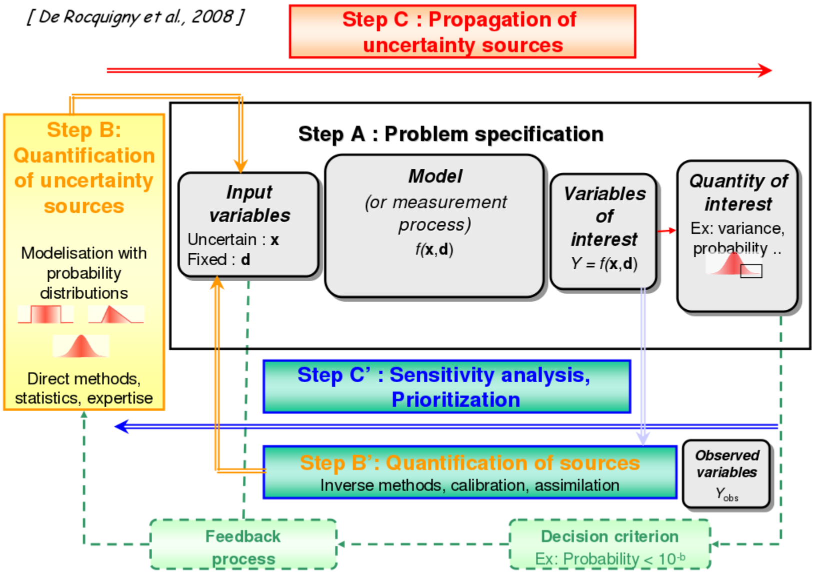

Figure: Uncertainty study and management in an industrial approach.

Source: Bertrand Iooss, ENBIS-EMSE 2009 Conference, Saint-Étienne, July 2009.

- Travaux de recherche au CERFACS[1], Toulouse
- Avec Paul Mycek[2] ainsi que Sophie Ricci[1,2], Mélanie Rochoux[1,2], Pamphile Roy[2] et Nicole Goutal[3]

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  2. Analyse de sensibilité (ex. : méthodes de Morris, Sobol', etc.)
  3. Réduction d'incertitudes (ex. : criblage, mise-à-jour, etc.)
  4. Assimilation de données (ex. : filtre de Kalman d'ensemble)

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- Deux types de solution :
  - Modèles de substitution
  - **Méthodes d'échantillonnage**

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Computer experimentation in hydraulics

Application: the Garonne river between Tonneins and La Réole

Framework: hydraulic network over the Garonne river

- length: 647 km,
- drained area: 55.000 km^2,
- a Garonne flood in 2009 at Marmande.

Computer experimentation in hydraulics

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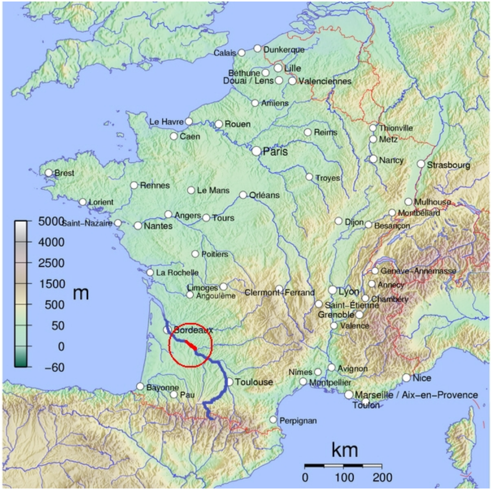

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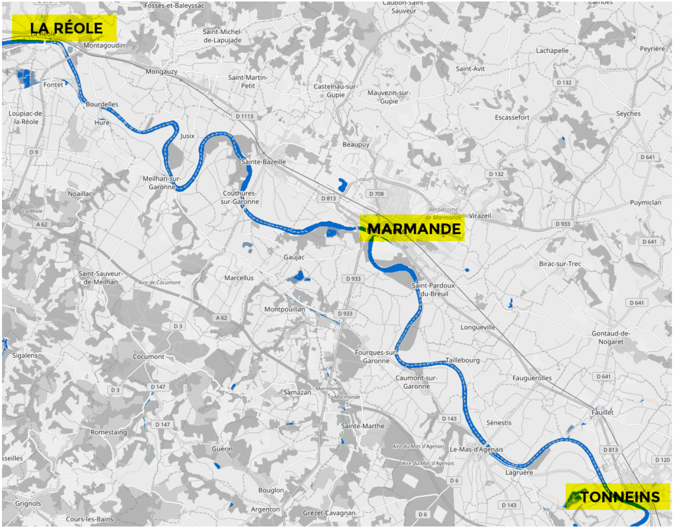

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Application: focus on the section Tonneins - La Réole

- from Tonneins ($s = 13$ km) to La Réole ($s = 62$ km),
- observation station at Marmande ($s = 13$ km),
- length: 50 km,
- mean slope: 3.3 m.km^{-1},
- mean width: 250 m,
- bank-full discharge $\approx$ mean annual discharge $= 1.000$ m^3.s^{-1}.

Computer experimentation in hydraulics

Discharge and water level, solutions of the shallow water equations

Navier-Stokes equations (NSE)

modeling the free surface flow dynamics with the hydraulic state:

$$\{Q(s,t) : \text{ discharge } [\mathrm{m}^3.\mathrm{s}^{-1}];\ h(s,t) : \text{water level } [\mathrm{m}]\}$$

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derived from NSE based on the following assumptions:

- height $\gg$ width,
- small bathymetry variations,
- hydrostatic pressure.

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$\Rightarrow$ Hyperbolic system of partial differential equations representing:

- subcritical flows (flow velocity $<$ wave velocity),
- supercritical flows (flow velocity $>$ wave velocity),
- hydraulic jumps (abrupt slowing down $\Rightarrow$ rather abrupt rise).

Computer experimentation in hydraulics
The software MASCARET in the field of free-surface flow

Welcome to TELEMAC-MASCARET

TELEMAC-MASCARET is an integrated suite of solvers for use in the field of free-surface flow. Having been used in the context of many studies throughout the world, it has become one of the major standards in its field.

TELEMAC-MASCARET is managed by a consortium of core organisations: **Artelia** (formerly Sogreah, France), **BundesAnstalt für Wasserbau** (BAW, Germany), **Centre d'Etudes et d'Expertise sur les Risques, l'Environnement, la Mobilité et l'Aménagement** (CEREMA, France), **Daresbury Laboratory** (United Kingdom), **Electricité de France R&D** (EDF, France), and **HR Wallingford** (United Kingdom).

TELEMAC-MASCARET is used by most partners for dimensioning and impact studies, where safety is prevailing and, for this reason, reliability, validation and a worldwide recognition of our tools are of utmost importance. As a consequence and to improve the access to TELEMAC-MASCARET for the whole community of consultants and researchers, the choice of open source has been made. Anyone can thus take advantage of TELEMAC-MASCARET and assess its performances, and will find necessary resources on this website. However the quality of assistance, maintenance and hotline support are also very important to professional users, and a special effort has been made to offer alternatively a broad range of fee-paying services.

Latest News

2017 User Conference - Registration now open !
14 August 2017

Registration for the upcoming XXIVth TELEMAC-MASCARET Users Conference is open ! You can register for the conference under www.tuc2017.tugraz.at where you will find also the preliminary program and t [...]

07 August 2017 **v7p2r2 is available for download**

31 July 2017 **Telemac Summer School 2017**

21 July 2017 **Fudaa-Mascaret 3.6 is available for Download**

Computer experimentation in hydraulics

Modelling water levels and flow rates for a river using MASCARET

Simulator: MASCARET software (using our Python API)

- simulates the one-dimensional shallow water equations,
- returns the hydraulic state (Q, h),
- over a discrete hydraulic network $\{s_{\text{in}}, s_{\text{in}} + \Delta s, \ldots, s_{\text{out}}\}$,
- at different times $\{t_1, t_1 + \Delta t, \ldots, t_T\}$.

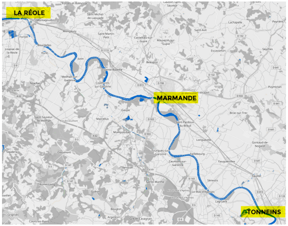
Applications: dam-break wave simulation, reservoir flushing and flooding, low flow, flood spreading, ...

MASCARET inputs

- bathymetry and roughness coefficients,
- upstream/downstream boundary conditions,
- lateral inflows,
- initial conditions.

Computer experimentation in hydraulics

Water level uncertainty quantification by Monte-Carlo methods



Uncertain parameters

- Strickler coefficient K_s,
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Quantities of interest

- water level at Marmande,
- even at different sites.

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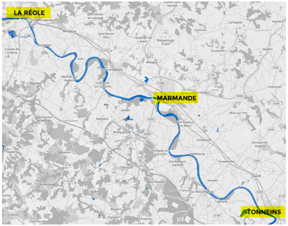

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- statistics (e.g. mean or variance),
- sensitivity indices (e.g. Sobol' indices),
- excess probabilities,
- covariance matrices for data assimilation,
- ...

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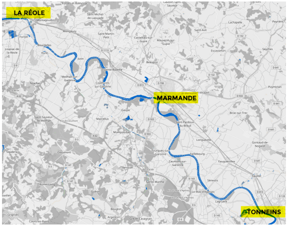

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Bottleneck

Variance of Monte-Carlo estimators *vs.* Simulator cost

Monte Carlo (MC) sampling

Implementation facility *vs.* computational cost

The physical phenomenon

$$f: \quad \mathcal{X} \to \mathbb{R}$$
$$\mathbf{x} \mapsto y = f(\mathbf{x})$$

The considered simulator

$$f_L: \quad \mathcal{X} \to \mathbb{R}$$
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where L denotes an accuracy level assumed satisfying.

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Uncertainty framework

Input parameters $\mathbf{x} = (x_1, \ldots, x_d)$ are uncertain:

- $\mathbf{x}$ is an instance of the r.v. $\mathbf{X} = (X_1, \ldots, X_d) \sim \mathcal{L}(X_1, \ldots, X_d)$

Thus, the outputs $y := f(\mathbf{x})$ and $y_L := f_L(\mathbf{x})$ are uncertain:

- y is an instance of the r.v. $Y = f(\mathbf{X}) \sim \mathcal{L}(Y) = ?$
- y_L is an instance of the r.v. $Y_L = f_L(\mathbf{X}) \sim \mathcal{L}(Y_L) = ?$

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MC estimation of mean, covariance, variance, ...

For any application $g : \mathcal{X} \to \mathbb{R}$ and any sample size N:

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$$\text{Mean: } \hat{m}_N\left[g(\mathbf{X})\right] = \frac{1}{N}\sum_{i=1}^{N} g\left(\mathbf{X}^{(i)}\right)$$

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$\Rightarrow$ Accuracy $\sim \mathcal{O}\left(N^{-1/2}\right)$ *vs.* computational cost $\sim \mathcal{O}\,$(m,h,d,w) .

Solution: The multi-level MC (MLMC)

e.g. MC expectation using codes with increasing accuracy and cost

Let $\{f_\ell\}_{\ell \geq 0}$ be a sequence of simulators with:

- $Y_\ell \equiv f_\ell(\mathbf{X})$ and $Y_{-1} \equiv 0$,

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Expressing the expectation as a telescoping sum

$$\mathbb{E}[Y_L] \quad = \quad \mathbb{E}[Y_0] \quad + \quad \textstyle\sum_{\ell=1}^{L} \quad \mathbb{E}[Y_\ell] \quad - \quad \mathbb{E}[Y_{\ell-1}]$$

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- $E_\ell[Y_{\ell'}] := \hat{m}_{N_\ell}[f_{\ell'}(\mathbf{X})] = \frac{1}{N_\ell} \sum_{i=1}^{N_\ell} Y_{\ell'}^{(\ell,i)} \approx \mathbb{E}[Y_{\ell'}]$
- $Y_{\ell'}^{(\ell,i)} \equiv f_{\ell'}\left(\mathbf{X}^{(\ell,i)}\right)$
- $\mathbf{X}^{(\ell,1)}, \ldots, \mathbf{X}^{(\ell,N_\ell)}$ a N_ℓ-sample with $\mathbf{X}^{(\ell,1)} \sim \mathcal{L}(\mathbf{X})$.

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Cost and variance of the MLMC estimator (Giles, 2008)

$$C \equiv \sum_{\ell=0}^{L} N_\ell C_\ell \qquad\qquad V \equiv \sum_{\ell=0}^{L} V_\ell / N_\ell$$

- $C_\ell =$ computational cost of $Y_\ell - Y_{\ell-1}$.
- $V_\ell =$ variance of $Y_\ell - Y_{\ell-1}$.

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$\forall$ accuracy ε s.t. $\mathbb{E}\left[\left(E_L^{\mathrm{ML}}[Y] - \mathbb{E}[Y]\right)^2\right] < \varepsilon^2$, there are $N_0, N_1, \ldots, N_\ell$ satisfying the optimal cost:

$$C = \varepsilon^{-2}\left(\sum_{\ell=0}^{L} \sqrt{V_\ell C_\ell}\right)^2 .$$

$\Rightarrow$ When $V_\ell C_\ell$ increases (resp. decreases), C is reduced by factor V_L/V_0 (resp. C_0/C_L) compared to standard MC with level L.

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Theorem (Giles, 2008)

Let $\{M_\ell\}_{\ell=0}^{\infty} \in \mathbb{N}_+$ *with an exponential increase s.t.* $M_\ell / M_{\ell-1} \geq a$ *for some fixed* a.

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Let $\{M_\ell\}_{\ell=0}^{\infty} \in \mathbb{N}_+$ *with an exponential increase s.t.* $M_\ell / M_{\ell-1} \geq a$ *for some fixed* a*. Let* α, β, γ *be positive constants s.t.:*

$$(i) \quad |\mathbb{E}[Y_\ell - Y_{\ell-1}]| \lesssim M_\ell^{-\alpha} \qquad (ii) \quad V_\ell \lesssim M_\ell^{-\beta} \qquad (iii) \quad C_\ell \lesssim M_\ell^{\gamma}.$$

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Then, $\forall \varepsilon > 0$, $\exists L$, $\exists \{N_\ell\}_{0 \leq \ell \leq L}$ s.t. $\mathbb{E}\left[\left(E_L^{ML} - \mathbb{E}[Y]\right)^2\right] < \varepsilon^2$, the overall computational cost verifies:

$$C \lesssim \varepsilon^{-\frac{\gamma}{\alpha}} + \varepsilon^{-2}\left(\mathbb{1}_{\beta > \gamma} + |\log(\varepsilon)|^2 \, \mathbb{1}_{\beta = \gamma} + \varepsilon^{-\frac{\gamma - \beta}{2}} \mathbb{1}_{\beta < \gamma}\right)$$

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$\Rightarrow$ Up to a logarithmic term, the computational cost is reduced by a factor of the order of $\varepsilon^{\frac{\min(2\alpha,\beta,\gamma)}{\alpha}}$ vs. standard MC with level L.

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Let $\{M_\ell\}_{\ell=0}^{\infty} \in \mathbb{N}_+$ *with an exponential increase s.t.* $M_\ell / M_{\ell-1} \geq a$ *for some fixed* a*. Let* α, β, γ *be positive constants s.t.:*

$$(i) \quad |\mathbb{E}\,[Y_\ell - Y_{\ell-1}]| \lesssim M_\ell^{-\alpha} \qquad (ii) \quad V_\ell \lesssim M_\ell^{-\beta} \qquad (iii) \quad C_\ell \lesssim M_\ell^{\gamma}.$$

Then, $\forall \varepsilon > 0$, $\exists L$, $\exists \{N_\ell\}_{0 \leq \ell \leq L}$ *s.t.* $\mathbb{E}\left[\left(E_L^{ML} - \mathbb{E}[Y]\right)^2\right] < \varepsilon^2$, *the overall computational cost verifies:*

$$C \lesssim \varepsilon^{-\frac{\gamma}{\alpha}} + \varepsilon^{-2}\left(\mathbb{1}_{\beta>\gamma} + |\log(\varepsilon)|^2\,\mathbb{1}_{\beta=\gamma} + \varepsilon^{-\frac{\gamma-\beta}{2}}\mathbb{1}_{\beta<\gamma}\right)$$

$\Rightarrow$ Up to a logarithmic term, the computational cost is reduced by a factor of the order of $\varepsilon^{\frac{\min(2\alpha,\beta,\gamma)}{\alpha}}$ vs. standard MC with level L.

Remark: M_ℓ represents the "complexity" of the software at level ℓ. For example, we can choose $M_\ell = 2^\ell$ when the size of the mesh differs by a factor of 2 between levels $\ell - 1$ and ℓ.

Efficient computation of MLMC expectation

Sequential selection of N_ℓ and L (Giles, 2015)

1. Initialization:
    - $L_{\min}$, $L_{\max}$ = minimal, maximal number of levels
    - $L := L_{\min}$ = current number of levels
    - $N_0^{[1]}, N_1^{[1]}, \ldots, N_L^{[1]}$ = small initial sample sizes

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   Else if accuracy not reached and $L < L_{\text{max}}$

   Add a finer level $L := L + 1$

   Initialize the variance and the cost $Y_L - Y_{L-1}$

   Update $N_{0,\text{opt}}, \ldots, N_{L,\text{opt}}$

   Back to Step 2. with $it := it + 1$

Extension of the MLMC approaches

to more general quantities (Bierig and Chernov, 2015)

MLMC estimator built from unbiased MC estimators

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- θ be the quantity to be estimated

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Then, we may defined the corresponding MLMC estimator:

$$\hat{\theta}_L^{\mathrm{ML}} = \sum_{\ell=0}^{L} \hat{\theta}_{\ell,\ell} - \hat{\theta}_{\ell-1,\ell}, \quad \hat{\theta}_{-1,0} \equiv 0,$$

with:

- $\mathbb{E}\left[\hat{\theta}_L^{\mathrm{ML}} - \theta_L\right] = 0,$
- $\mathbb{E}\left[\hat{\theta}_L^{\mathrm{ML}} - \theta\right] = \theta_L - \theta \xrightarrow{L \to \infty} 0.$

General MLMC framework

MLMC estimators of covariance and variance

For uncertainty quantification and data assimilation purposes, we are particularly interested in estimating variances and covariances.

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Unbiased covariance MLMC estimator

Let Y and Z be two r.v. Then, we have:

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where $C_\ell[Y_{\ell'}, Z_{\ell'}] = \frac{N_\ell}{N_\ell - 1} E_\ell\left[(Y_{\ell'} - E_\ell[Y_{\ell'}])(Z_{\ell'} - E_\ell[Z_{\ell'}])\right]$.

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Special case: the unbiased variance MLMC estimator

$$V_L^{\mathrm{ML}}[Y] \equiv C_L^{\mathrm{ML}}[Y, Y] = \sum_{\ell=0}^{L} V_\ell[Y_\ell] - V_\ell[Y_{\ell-1}]$$

where $V_\ell[Y_{\ell'}] \equiv C_\ell[Y_{\ell'}, Y_{\ell'}]$.

General MLMC framework

Mean square error decomposition and bias/variance trade-off

Mean square error (MSE) of a MLMC estimator

$$\mathsf{MSE}\left(\hat{\theta}_L^{\mathrm{ML}}, \theta\right) \equiv \mathbb{E}\left[\left(\hat{\theta}_L^{\mathrm{ML}} - \theta\right)^2\right]$$

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&= \mathbb{V}\left[\hat{\theta}_L^{\mathrm{ML}}\right] + \mathsf{Bias}\left(\hat{\theta}_L^{\mathrm{ML}}, \theta\right)^2
\end{aligned}$$

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&= \underbrace{\mathsf{MSE}\left(\hat{\theta}_L^{\mathrm{ML}}, \theta_L\right)}_{\substack{\text{sampling error} \\ \text{COST}}} + \underbrace{|\theta_L - \theta|^2}_{\substack{\text{discretization error} \\ \text{ACCURACY}}} \quad \text{if } \hat{\theta}_L^{\mathrm{ML}} \text{ is unbiased}
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$$

Objective

Controlling the MSE of the MLMC estimator $\hat{\theta}_L^{\mathsf{ML}}$ using a sequence of number of evaluations $\{N_\ell\}_{0 \le \ell \le L}$ balancing:

- its cost ($=$ sampling error)
- its accuracy ($=$ discretization error).

General MLMC framework

Bounding the variance of a MLMC estimator[2]

$\hat{\theta}_L^{\mathrm{ML}} = \ldots$	$\mathbb{V}\left[\hat{\theta}_L^{\mathrm{ML}}\right] \leq \ldots$
$E_L^{\mathrm{ML}}[Y]$	$\sum_{\ell \leq L} \frac{1}{N_\ell} \mathbb{V}[\Delta_\ell^Y]$ (equality)
$V_L^{\mathrm{ML}}[Y]$	$\sum_{\ell \leq L} \frac{1}{N_\ell - 1} \sqrt{\mathbb{M}^4[\Delta_\ell^Y]\, \mathbb{M}^4[\Sigma_\ell^Y]}$

where:

- $\Delta_\ell^Y \equiv Y_\ell - Y_{\ell-1}$,
- $\Sigma_\ell^Y \equiv Y_\ell + Y_{\ell-1}$,
- $\mathbb{M}^4$ denotes the fourth central moment.

[2] For $E_L^{\mathrm{ML}}[Y]$, see Giles, 2008. For $V_L^{\mathrm{ML}}[Y]$, see Bierig and Chernov, 2015.

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Estimating a covariance by MLMC methods

Extending the results of Bierig and Chernov, 2015

Theorem

Let Y and Z be r.v. Let Y_ℓ and Z_ℓ be their approximations at level ℓ.

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Let $(M_\ell)_{\ell\geq 0}$ be an exponentially increasing sequence of positive integers satisfying $M_\ell/M_{\ell-1} \geq a$ for some fixed $a > 1$.
Moreover, assume that there exist constants $\alpha, \beta, \gamma > 0$ s.t.:

$$\text{(i) } |\mathbb{C}[Y_\ell, Z_\ell] - \mathbb{C}[Y, Z]| \lesssim M_\ell^{-\alpha},$$
$$\text{(ii) } \sqrt{\mathbb{M}^4[\Delta_\ell^Y]} + \sqrt{\mathbb{M}^4[\Delta_\ell^Z]} \lesssim M_\ell^{-\beta}, \text{ (iii) } \mathcal{C}_\ell \lesssim M_\ell^{\gamma},$$

and assume that $\mathbb{M}^4[Y_\ell]$ and $\mathbb{M}^4[Z_\ell]$ are uniformly bounded.

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and assume that $\mathbb{M}^4[Y_\ell]$ and $\mathbb{M}^4[Z_\ell]$ are uniformly bounded.
Then for any $0<\varepsilon<e^{-1}$, there exist an integer L and a sequence of positive integers $(N_\ell)_{\ell\geq 0}$ s.t. $\mathbb{E}\left[(C_L^{\mathrm{ML}}[Y,Z]-\mathbb{C}[Y,Z])^2\right]<\varepsilon^2$ and:

$$\mathrm{Cost}_\varepsilon\left(C_L^{\mathrm{ML}}[Y,Z]\right)\lesssim \varepsilon^{-\frac{\gamma}{\alpha}}+\varepsilon^{-2}\left(\mathbb{1}_{\beta>\gamma}+\log(\varepsilon)^2\mathbb{1}_{\beta=\gamma}+\varepsilon^{-\frac{\gamma-\beta}{2}}\mathbb{1}_{\beta<\gamma}\right).$$

Numerical experiments for MLMC covariance

Study case: a stochastic differential equation - Presentation

An ordinary differential equation with a random coefficient

$$\begin{cases} \dfrac{\mathrm{d}u(t,\omega)}{\mathrm{d}t} = a(\omega)u(t,\omega), \quad t \in (0,1], \quad a \sim \mathcal{N}(\mu,\sigma^2) \\ u(0,\cdot) = U_0, \end{cases}$$

whose solution $u(t,\cdot) = U_0 e^{at} \sim \ln \mathcal{N}(\mu t + \ln U_0, \sigma^2 t^2)$ is a log-normal process.

Statistics with analytical formulation

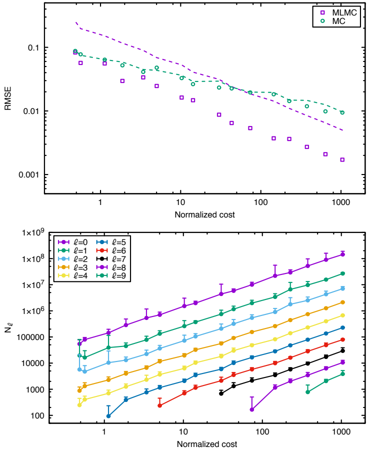
$\mathbb{E}[u(t,\cdot)]$, $\mathbb{V}[u(t,\cdot)]$, $\mathbb{C}[u(t_1,\cdot),u(t_2,\cdot)]$, $\mathbb{C}[a,u(t,\cdot)]$, $\mathbb{M}^4[u(t,\cdot)]$ and $\mathbb{M}^4[a]$.

Definition of the different levels

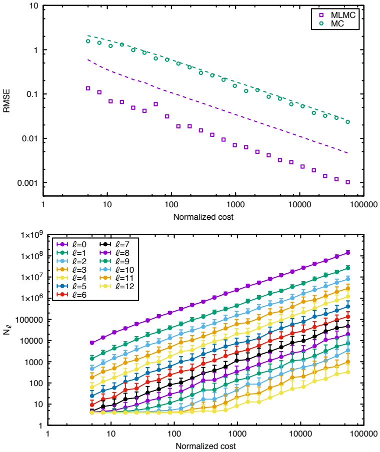
$M_\ell = 16 \times 2^\ell$ esquispaced points at level ℓ

Numerical experiments for MLMC covariance

Study case: a stochastic differential equation - Results



Target: ACCURACY



Target: COST

Extending the MLMC techniques

to data assimilation

Ensemble Kalman filtering

- MLMC covariance matrices based:
  - either on a direct application of the MLMC covariance, using a worst-case strategy and sparse matrices
  - either on an extension of the theorem dedicated to the covariance, using a MSE definition dedicated to covariance matrices.
- MLMC Ensemble Kalman filtering (see Hoel et al, 2016)

Extending the MLMC techniques

to sensitivity analysis

Sobol' index $S(X_i) = \mathbb{V}[\mathbb{E}[Y|X_i]]/\mathbb{V}[Y]$

1. $S(X_i) \propto \mathbb{V}[\mathbb{E}[Y|X_i]] \leftarrow$ trick for MLMC + dimensioned value

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3. Apply MLMC techniques for the covariance.

Extension of Janon, 2013

Let $\left\{S^{(\ell)}_{i,\Psi_\ell(N)}\right\}_{0\leq\ell\leq L}$ be a sequence of pick-and-freeze estimators of the Sobol' index $S(X_i)$ where $\Psi_\ell : \mathbb{N}^* \to \mathbb{N}^*$ and $\Psi_0(N) > \Psi_1(N) > \ldots > \Psi_L(N) \equiv N$.
Then, we define the multi-fidelity estimator:

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Then, we define the multi-fidelity estimator:

$$S^{\mathrm{ML}}_{i,N} = \sum_{\ell=0}^{L} S^{(\ell)}_{i,\Psi_\ell(N)} - S^{(\ell-1)}_{i,\Psi_\ell(N)}$$

Extending the MLMC techniques

to sensitivity analysis

Theorem (Concentration inequality; extension of Janon, 2013))

If $\forall \ell \in \{0, 1, \ldots, L\}$, $\lim_{N \to \infty} \Psi_\ell(N) = \infty$,
Then, for any sequence of accuracy levels $\{\alpha_\ell\}_{0 \le \ell \le L}$ *in* $]0, 1[$, *we have:*

$$\lim_{N \to \infty} \mathbb{P}\left[\left|S_{i,N}^{ML} - S_i\right| \le \sum_{\ell=0}^{L} q(\alpha_\ell) \frac{\sigma_\ell}{\sqrt{\Psi_\ell(N)}}\right] \ge 1 - \sum_{\ell=0}^{L} \alpha_\ell$$

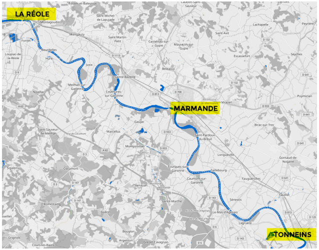
with $q(a) = \Phi^{-1}(1 - a/2)$

$$\begin{cases} \sigma_0 = \frac{\mathbb{V}[A_0 - B_0/2]}{(\mathbb{V}[Y_0])^2} \\ \sigma_\ell = \sigma_{\ell-1} + \frac{\mathbb{V}[A_\ell - B_\ell/2]}{(\mathbb{V}[Y_\ell])^2} - \frac{2\,\mathrm{Cov}[A_\ell, A_{\ell-1}] - \big(\mathrm{Cov}(A_\ell, B_{\ell-1}) + \mathrm{Cov}(B_\ell, A_{\ell-1})\big) + \mathrm{Cov}(B_\ell, B_{\ell-1})/2}{\mathbb{V}[Y_\ell]\mathbb{V}[Y_{\ell-1}]}, \ \ell > 0 \end{cases}$$

with $A_\ell = (Y_\ell - \mathbb{E}[Y_\ell])(Y_\ell^{\{i\}} - \mathbb{E}[Y_\ell])$ *and* $B_\ell = (Y_\ell - \mathbb{E}[Y_\ell])^2 + (Y_\ell^{\{i\}} - \mathbb{E}[Y_\ell])^2$.

Application of MLMC in hydraulic modeling

Coming back to the Garonne river



Output of interest:

- Water level at Marmande h

Uncertain inputs:

- Strickler coefficient K_s
  $K_s \sim \mathcal{N}(35, 3.6)$
- Upstream flow Q
  $Q \sim \mathcal{N}(1500, 102)$

MLMC parameters:

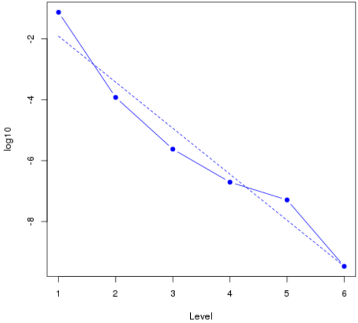
- Levels:
  $\ell \in \{0, 1, \ldots, L_{\max}\}$
- Finest level: $L_{\max} = 6$
- Number of nodes by level:
  - $N_{x,0} = 25$ nodes
  - $N_{x,\ell} = 2^\ell \times N_0$ nodes

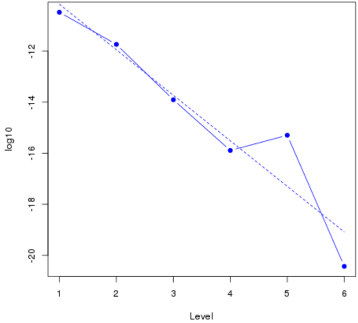
CPU time$(Y_\ell) \sim \mathcal{O}(2^{0.98\ell})$:

0. 0.2 s
1. 0.3 s
2. 0.6 s
3. 1.1 s
4. 2.3 s
5. 4.5 s
6. 8.9 s

Application in hydraulic modeling

Results concerning the mean of the water level





$$|\mathrm{Mean}\,(Y_\ell - Y_{\ell-1})| \sim \mathcal{O}\left(2^{-\alpha\ell}\right)$$

$$\alpha = 1.21$$

$$|\mathrm{Variance}\,(Y_\ell - Y_{\ell-1})| \sim \mathcal{O}\left(2^{-\beta\ell}\right)$$

$$\beta = 1.90$$

Application in hydraulic modeling

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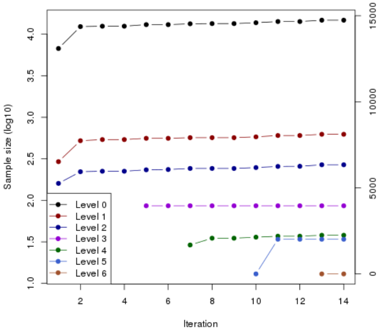
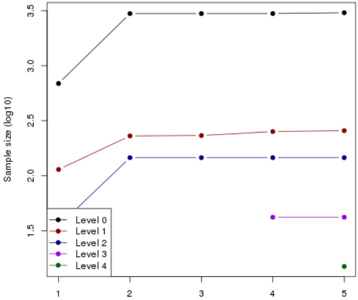


Accuracy: $\varepsilon = 5.000e-03$

- Number of runs:
  - Level 0: 14738 runs
  - Level 1: 624 runs
  - Level 2: 268 runs
  - Level 3: 86 runs
  - Level 4: 38 runs
  - Level 5: 34 runs
  - Level 6: 13 runs
- Comparison at $\ell = 6$:
  - MLMC $\approx$ 317 runs
  - MC $\approx$ 11482 runs
- **Savings: 36.24**

Application in hydraulic modeling

Results concerning the variance of the water level

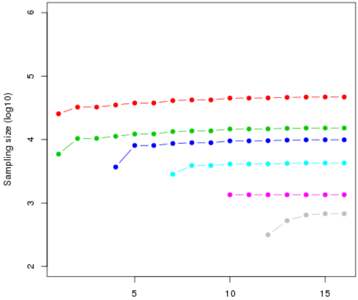


Accuracy: $\varepsilon = 1.000e - 02$

- Number of runs:
  - Level 0: 3030 runs
  - Level 1: 257 runs
  - Level 2: 146 runs
  - Level 3: 42 runs
  - Level 4: 15 runs
- Comparison at $\ell = 4$:
  - MLMC $\approx$ 294 runs
  - MC $\approx$ 11482 runs
- **Savings: 4.39**

Application in hydraulic modeling

Results concerning a Sobol' index



Accuracy: $\varepsilon = 1.000e - 03$

- Number of runs:
  - Level 0: 341489 runs
  - Level 1: 46837 runs
  - Level 2: 15202 runs
  - Level 3: 9915 runs
  - Level 4: 4273 runs
  - Level 5: 1347 runs
  - Level 6: 677 runs
- Comparison at $\ell = 6$:
  - MLMC $\approx$ 11407 runs
  - MC $\approx$ 97651 runs
- **Savings: 8.56**

Conclusion

- General framework for MLMC approach
- Extension to the covariance case:
  - Theorem
  - Adaptative algorithm
- Practical extension to the target "cost" (adaptative algorithm)
- Successfully applied to an analytical test case
  - Expectation, variance, covariance
  - Comparison of the computed and theoretical bounds
- Successfully applied to an hydraulic problem
  - Expectation, variance, covariance, numerator of Sobol' indices
  - Important reduction of the computation cost
- Straightforward extension of the multi-fidelity estimator for Sobol' indices

Perspectives

- Publication: work in progress :-)
- Extension of the MLMC covariance estimator to the case of covariance matrices
  - Data assimilation
- Extension of the MLMC pick-and-freeze Sobol' index to the case of a set of MLMC pick-and-freeze Sobol' indices
  - Sensitivity analysis
- Application of the MLMC methods to an hydraulic problem based on an unstructured mesh
- MLMC strategy for the construction of multifidelity surrogate models (e.g. cokriging)
- MLMC strategy for excess probabilities

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Ali Baba's cave: `https://people.maths.ox.ac.uk/gilesm/mlmc_community.html`

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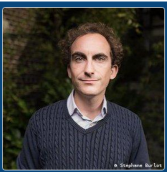

Qui suis-je ?

Matthias DE LOZZO

30 ans, originaire de Toulouse
Actuellement à Toulouse, après une expérience de deux ans en Provence

Mon profil LinkedIn - Mon profil ResearchGate

Ingénieur-Docteur en Modélisation Statistique

> Formation : INSA de Toulouse (ingénieur depuis 2010 - docteur depuis 2013)

> Depuis octobre 2016 : post-doctorant au CERFACS (Centre Européen de Recherche et de Formation Avancée en Calcul Scientifique)
> Avril 2014 - Mars 2016 : post-doctorant au CEA (Commissariat à l'énergie atomique et aux énergies alternatives) de Cadarache

> Qualification (MCF) - Section 26 (Mathématiques appliquées et applications des mathématiques)

> Expérience professionnelle : CNRM - Météo France, ONERA, Epsilon-Ingénierie, CEA puis CERFACS.

> Attrait pour un large champ d'application des statistiques : environnement, énergie, économie, social, industrie, santé, *text mining, big data* ...

Métier

Compétences :

- Apprentissage statistique
- Exploration de données
- *Machine learning & Data mining*
- Modèles de régression et de décision
- Modèles mixtes, durée de vie
- Plans d'expériences et enrichissement
- Tests statistiques
- ...

Thématiques de recherche :

- Gestion des incertitudes
- Analyse de sensibilité
- Modèles de substitution
- Métamodèles par processus gaussien

Informatique

Métier : LaTeX ♦ Matlab ♦ OpenTURNS ♦ R ♦ R Shiny ♦ SAS

Programmation : C ♦ C++ ♦ Fortran ♦ Python

Web : HTML ♦ PHP ♦ CSS ♦ SQL

Système : Windows ♦ Linux

Langues

Français : langue maternelle

Anglais : professionnel

Espagnol : intermédiaire

Centres d'intérêt

Associatif ♦ *Data analysis* et *Data visualization* ♦ Environnement et Société ♦ Montagne ♦ *Webmastering*

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COMING SOON!

2017 **Comparison of Polynomial Chaos and Gaussian Process surrogates for uncertainty quantification and correlation estimation of spatially distributed open-channel steady flows**, P. T. Roy, N. El Moçayd, S. Ricci, J.-C. Jouhaud, N. Goutal, M. De Lozzo, M. Rochoux, soumis à Stochastic Environmental Research and Risk Assessment.

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2013 **Multilayer perceptron for the learning of spatio-temporal dynamics - Application in thermal engineering**, M. De Lozzo, P. Klotz, B. Laurent, Engineering Applications of Artificial Intelligence, 26(10), p. 2270–2286.

Jounaux français

2015 **Substitution de modèle et approche multifidélité en expérimentation numérique**, M. De Lozzo, Journal de la Société Française de Statistique, 156(3), p. 21–55.

Actes de conférences

2017 **Méthodes de Monte-Carlo multi-niveaux pour la quantification d'incertitudes et l'assimilation de données - Application à la modélisation fluviale**, M. De Lozzo, P. Mycek, S. Ricci, M. Rochoux, P. T. Roy, N. Goutal, 49èmes Journées de Statistique, SFDS, Avignon, du 29 mai au 2 juin 2017.

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2015 **Uncertainty propagation for reliability analysis: application to sodium fast reactor safety studies**, A. Marrel, N. Seiler-Marie, M. De Lozzo, UNCECOMP, Hersonissos (Grèce), du 25 au 27 mai 2015.

2012 **Learning of Spatio-temporal Dynamics in Thermal Engineering**, M. De Lozzo, P. Klotz, B. Laurent, Engineering Applications of Neural Networks dans Communications in Computer and Information Science, 311, p. 213–222.

MERCI
pour votre attention :-)

Matthias De Lozzo

Plus d'informations sur
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Assimilation de données par filtre de Kalman

Modèle d'observation

$$\mathbf{Y}^o = \mathbf{G}\mathbf{X}^b + \varepsilon$$

où $\mathbf{X}^b \sim \mathcal{N}\left[\mathbf{X}^t, \mathbf{B}\right],\ \varepsilon \perp \mathbf{X}^b$ et $\varepsilon \sim \mathcal{N}\left[\mathbf{0}, \boldsymbol{R}\right]$

$$\downarrow$$

Vecteur gaussien de l'observation et de l'analyse

$$\begin{pmatrix} \mathbf{X}^a \\ \mathbf{Y}^o \end{pmatrix} \sim \mathcal{N}\left[\begin{pmatrix} \mathbf{x}^b \\ \mathbf{G}\mathbf{x}^b \end{pmatrix}, \begin{pmatrix} \mathbf{B} & \mathbf{B}\mathbf{G}^T \\ \mathbf{G}\mathbf{B} & \mathbf{G}\mathbf{B}\mathbf{G}^T + \mathbf{R} \end{pmatrix}\right]$$

$$\downarrow$$

Estimation de l'analyse (accompagnée d'une mesure d'erreur)

$$\mathbf{x}^a = \mathbb{E}[\mathbf{X}^a | \mathbf{Y}^o = \mathbf{y}^o]$$

$$= \mathbf{x}^b + \mathbf{B}\mathbf{G}^T \left(\mathbf{G}\mathbf{B}\mathbf{G}^T + \mathbf{R}\right)^{-1} \left(\mathbf{y}^o - \mathbf{G}\mathbf{x}^b\right)$$