Statistics and learning

Tests

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Motivations

When could tests be useful ?

- A statistical hypothesis is an assumption on the distribution of a random variable.
- Ex: test whether the average temperature in a holiday ressort is $28°\mathrm{C}$ in the summer.
- A test is a procedure which makes use of a sample to decide whether we can reject an hypothesis or whether there is nothing wrong with it (it's not really acceptance).
- Examples of applications: decide if a new drug can be put on market after adequate clinical trials, decide if items comply with predefined standards, which genes are significantly differentially expressed in pathological cells ...
- Typically, sources to build hypothesis stem from quality need, values from a previous experiment, a theory that need experimental confirmation or an assumption based on observations.

Outline and a motivating example

It's really about **decision making**; don't be fooled; tests shed light on a question, final results heavily depend on a human interpretation !

Today's goals:

- introduce basic concepts related to tests through 2 examples.
- A general presentation of tests.
- Some particular cases: one-sample, two-sample, paired tests; Z-tests, t-tests, χ^2-tests, F-tests...

Example 1: cheater detection

To introduce randomness, you are asked to throw a coin 200 times and write down the results. Why would I be suspicious about students that do not exhibit at least one HHHHHH or TTTTTT pattern ? Would I be (totally ?) fair if I was to blame (all of) them ?

Motivation 2

Example 2: rain makers

In a given area of agricultural interest, it usually rains 600 mm a year.

Suspicious scientists claim that they can locally increase rainfall, when spreading a revolutionary chemical (iodised silver) on clouds. Tests over the 1995-2002 period gave te following results:

Year	1995	1996	1997	1998	1999	2000	2001	2002
Rainfall (mm/year)	606	592	639	598	614	607	616	586

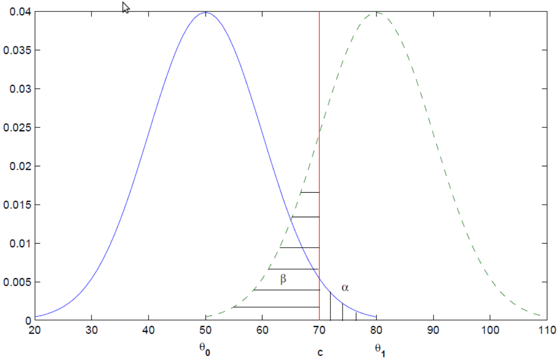
Does this sound correct to you ? Quantify the answer.

Bonus: what would have changed if you wanted to test if the increase was of say 30 mm ?

Motivation

Rain makers and possible errors

If you assume normality of rainfalls, had you applied the treatment or not



Hypothesis testing: (H0) $\theta = \theta_0$ and (H1) $\theta = \theta_1$.

Tests

Possible situations

		Realworld	
		(H0)	(H1)
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Apply that to 'innoncent until proven guilty' and interpret the different situations. How do you want to control α and β ?

What about introducing a new drug on the market ??

Tests

General methodology

1. Modelling of the problem.
2. Determine alternative hypotheses to test (disjoint but not necessarily exhaustive).
3. Choose of a statistic which (a) can be computed from data and (b) which has a known distribution under (H0).
4. Determine the behaviour of statistics under (H1) and build critical region (where (H0) rejected)
5. Compute the region at a fixed error I threshold and compare to values obtained from data. Or compute p-value of the test from data.
6. Statistical conclusion: accept or reject (H0). Comment on p-value ?

opt. Can you say something about the power ?

7. Strategic conclusion: how do YOU decide thanks to the light shed by statistical result ?

Test methodology into details

- **Hypothesis**:= any subset of the family of all considered probability distributions $\mathcal{P}$. In practice, hypotheses are often on unknown parameters of distributions $\rightarrow$ parametric hypotheses, defined by equalities or inequalities: (H0) $\theta_0 \in \Theta_0$ and (H1) $\theta_1 \in \Theta_1$. In turn, they can be simple if only one value for the parameters is tested or multiple composite.

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- Choose a **test statistic** $T_n :=$ a random variable which only depends on $(\Theta_0; \Theta_1)$ and on observations of the (X_i)'s. Interesting if the distribution is known given (H0) is true. Note that it is an estimator...depending on (H0) and (H1).

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- How to choose a good test statistic ? Remember the typology of confidence intervals ? And explore R help ?!

Test methodology into details (cont'd)

- Determine the **rejection region** R. Usually of the form $(r; +\infty)$, $(-\infty; r)$ or $(-\infty; r) \cup (r'; +\infty)$. To decide, examine how the test statistic behaves under (H1).

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- **p-value**:= maximal value of α so that the test would accept the observed statistic to be drawn under (H0) $\approx$ credibility index on (H0). Alternative definition: probability to obtain a test statistic value at least as contradictory to (H0) as the observed value assuming (H0) is true (if we repeated the experiment a large number of times).

Test methodology into details (end)

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- hence (H0) is chosen according to a firmly established theory (you don't want to make a fool of yourself), because caution is needed or...for subjective reasons (consumer choice is not that of manufacturers !)

Choosing hypotheses: launching a new drug

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Historical note: statistics were of great help in modern medicine.

Tests you need to know

and we shall see during next session and use on practical examples

- Parametric tests (observations drawn from $\mathcal{N}$ or large samples so that C.L.Th. applies)
  - one sample: comparing the empirical mean to a theoretical value $\rightarrow$ Z-test or t-test
  - two independent samples $\rightarrow$ t-test, F-test
  - paired samples $\rightarrow$ paired t-test
  - several samples $\rightarrow$ ANOVA (not today).
- Adequation tests $\rightarrow$ χ^2-tests. Normality check $\rightarrow$ Kolmogorov or Shapiro-Wilks.
- Non-parametric tests (when small samples or non Gaussian distributions)
  - comparing 2 medians from independent samples $\rightarrow$ Mann-Whitney test.
  - two paired samples $\rightarrow$ Wilcoxon test on differences.
  - several samples $\rightarrow$ Kruskal-Wallis.

Exercises

Poisson arrival at a motorway toll booth

For two hours, at a motorway toll, we write down the number of cars arriving during each 2 minute intervals. We obtain:

#(cars)	0	1	2	3	4	5	6	7	8	9	10	11
#(intervals)	4	9	24	25	22	18	6	5	3	2	1	1

Test at a significance level of 0.1 the fit to a Poisson distribution with a parameter to be determined.

Evolution of purchasing power

In 2004, the total amount spent on products which are not essentials (*e.g.* travels, shows, *etc.* as opposed to food, hoosing, *etc.*) was 632 euros per month per household accoring to the INSEE during a partial survey over millions of households. In 2008, from a sample of $2,000$ interviewed by telephone, $1,837$ answers were obtained and the declared mean value was 598 euros (with sd 254 euros). If you assume a 2% inflation per year, would you say that the amount spent on non-essentials has significantly decreased ?

Finished

Next time: more tests and analysis of variance (ANOVA)