Centre for
Data Analytics

Insight

Multi-Objective Constraint Optimization with Lexicographic *and Weighted Average* Preference Models

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Outline

- Introduction MOCOP
- Order Relations and Tradeoffs
- Computation of Preference Inference
- Solving MOCOP via AND/OR branch and bound
- Experiments

Background

- Real-world decisions often involve multiple, conflicting and non-commensurate objectives.
  - Oil drilling programmes:

PROFIT vs. ENVIRONMENTAL DAMAGE

MOCOP

A Multi-Objective Constraint Optimization Problem (MOCOP) instance consists of

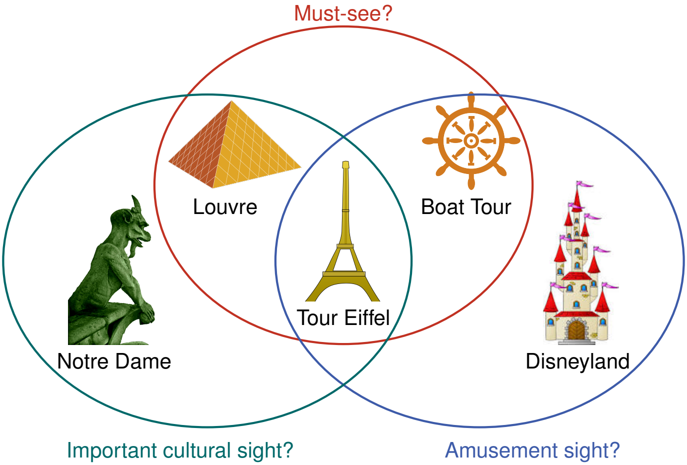
- Variables $\mathbf{X} = \{X_1, \ldots, X_n\}$ over
- Finite Domains $\mathbf{D} = \{D_1, \ldots, D_n\}$ and a set of
- Utility Functions $\mathbf{F} = \{f_1, \ldots, f_r\}$, with $f_i : \overline{Y_i} \longrightarrow \mathbb{R}^p$, where $Y_i \subseteq \mathbf{X}$ and the
- Number of Objectives is p.

Maximize
$\mathcal{F}(\mathbf{X}) = \sum_{i=1}^{r} f_i(Y_i)$.

MOCOP - Example

Paris Trip:

Two school classes A and B vote over attractions to see on their joint trip to Paris regarding 3 aspects.



MOCOP - Example

Paris Trip:

- Sights: $X_0, \dots, X_4$
- Aspects:

$$\begin{array}{lll} \text{Culture} & f_1 : \{X_0, X_1, X_2\} \searrow & \\ \text{Must-see} & f_2 : \{X_0, X_1, X_3\} \rightarrow & \text{(Votes A, Votes B)} \\ \text{Amusement} & f_3 : \{X_1, X_3, X_4\} \nearrow & \end{array}$$

X_0	X_1	X_2	f_1
0	0	0	(5,3)
0	0	1	(2,8)
0	1	0	(3,6)
0	1	1	(2,9)
1	0	0	(2,1)
1	0	1	(3,0)
1	1	0	(1,6)
1	1	1	(2,4)

X_0	X_1	X_3	f_2
0	0	0	(1,5)
0	0	1	(4,2)
0	1	0	(0,6)
0	1	1	(2,4)
1	0	0	(6,1)
1	0	1	(5,5)
1	1	0	(6,4)
1	1	1	(5,2)

X_1	X_3	X_4	f_3
0	0	0	(1,4)
0	0	1	(3,2)
0	1	0	(5,1)
0	1	1	(4,3)
1	0	0	(1,9)
1	0	1	(3,2)
1	1	0	(5,6)
1	1	1	(4,8)

$\rightarrow$ Maximize $\mathcal{F}(X) = f_1 + f_2 + f_3$

MOCOP

Maximize $\mathcal{F}(X) = f_1 + f_2 + f_3$

... but which vectors are maximal?

Compare utility vectors by

- Pareto order
- Weighted averages
- Lexicographic order
- ...

$\rightarrow$ Set of undominated solutions is “maximal”!

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Issues with Order Relations

Pareto order

- No cutting of relevant solutions for decision maker.
- Very large number of optimal (undominated) solutions.

Issues with Order Relations

Mapping all objectives into a single scale of utility

- Precise representation of objective.
- Time-consuming/Difficult to elicit trade-offs.
- More than one expert may be involved in decision making process.

Idea - Use Imprecise Tradeoffs

Elicit only a few tradeoffs between the objectives and deduce more information.

Example

In a bi-objective case, user may reveal that:

- $(1, 0)$ is at least as good as $(0, 1)$
- $(0, 2)$ is strictly preferred to $(1, 0)$

Idea - Use Imprecise Tradeoffs

Non-strict Statement	$\alpha \geq \beta$: "α is at least as good as β".
Strict Statement	$\alpha > \beta$: "α is strictly preferred to β".
Sets of Statements	$\Gamma_{\geq}$ the set of non-strict statements.
	$\Gamma_{>}$ the set of strict statements.
	$\Gamma = \Gamma_{\geq} \cup \Gamma_{>}$.

Order Relations

Weighted Average Models (WA):

- Normalized weights vector $\vec{w} = (w_1, \ldots, w_p)$ with $\sum_{i=1,\ldots,p} w_i = 1$.
- $\alpha \geq_w \beta$ if and only if $\sum_{i=1,\ldots,p} w_i * \alpha_i \geq \sum_{i=1,\ldots,p} w_i * \beta_i$ ($\vec{w}$ satisfies $\alpha \geq \beta$).

Example

$\vec{w} = (0.3, 0.7)$. Then $(0, 4) \geq_w (5, 1)$, because:

$$
\begin{array}{ll}
 & 2.8 = 0.3 * 0 + 0.7 * 4 \\
\geq & \\
 & 2.2 \quad = 0.3 * 5 + 0.7 * 1.
\end{array}
$$

Order Relations

Lexicographic Models (Lex):

- Ordered subset of objectives $\pi = (\pi_1, \ldots, \pi_r)$ with $\{\pi_1, \ldots, \pi_r\} \subseteq \{1, \ldots, p\}$.
- $\alpha \geq_\pi \beta$ if and only if $(\alpha_{\pi_1}, \ldots, \alpha_{\pi_r}) \geq_{lex} (\beta_{\pi_1}, \ldots, \beta_{\pi_r})$ (π satisfies $\alpha \geq \beta$).

Example

$\pi = (3, 1, 4)$. Then $(0,4,1,2) \leq (5,3,1,0)$, because:

$$(1,\ 0,\ 2) \leq_{Lex} (1,\ 5,\ 0).$$

Tradeoffs

Consistency

WA-consistency Set of tradeoffs Γ is WA-consistent, if there exists WA model $\vec{w}$ that satisfies all tradeoffs.

Lex-consistency Set of tradeoffs Γ is Lex-consistent, if there exists Lex model π that satisfies all tradeoffs.

$\rightarrow$ If Γ is Lex-consistent then Γ is WA-consistent.

Order Relations

Inference

$\Gamma \vDash \alpha \geq_{WA} \beta$ if and only if all WA models that satisfy all elicited tradeoffs Γ also satisfy $\alpha \geq \beta$.

$\Gamma \vDash \alpha \geq_{Lex} \beta$ if and only if all Lex models that satisfy all elicited tradeoffs Γ also satisfy $\alpha \geq \beta$.

$\rightarrow$ If $\Gamma \vDash \alpha \geq_{WA} \beta$ then $\Gamma \vDash \alpha \geq_{Lex} \beta$.
If $\Gamma \vDash \alpha >_{WA} \beta$ then $\Gamma \vDash \alpha >_{Lex} \beta$.

Relationship Between Weighted Average and Lex Models

Induced Relations:

$\alpha \geq_{\Gamma}^{WA} \beta$ if and only if $\Gamma \models \alpha \geq_{WA} \beta$.

$\alpha >_{\Gamma}^{WA} \beta$ if and only if $\Gamma \models \alpha \geq_{WA} \beta$ and $\Gamma \not\models \beta \geq_{WA} \alpha$.

$\alpha \geq_{\Gamma}^{Lex} \beta$ and $\alpha >_{\Gamma}^{Lex} \beta$ analogously.

$Opt(S, \succ)$ the set of $\succ$-undominated solutions in S.

- For non-strict relations, Lex is stronger than WA:
  $Opt(S, \geq_{\Gamma}^{Lex}) \subseteq Opt(S, \geq_{\Gamma}^{WA})$
- For strict relations, not necessarily the case:
  $Opt(S, >_{\Gamma}^{Lex}) \not\subseteq Opt(S, >_{\Gamma}^{WA})$

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Computation of weighted average model inferences

- For WA-consistent Γ and $\alpha, \beta \in \mathbb{R}^p$, to check if α is preferred to β use LP formulation [Marinescu et al., CP'13]:
  - Define set of non-strict linear inequalities involving α and β and real-valued variables.
  - Check satisfiability of inequalities with standard LP solver.
  - Return TRUE if and only if inequalities are satisfiable.

Computation of lexicographic model inferences

- For Lex-consistent Γ and $\alpha, \beta \in \mathbb{R}^p$, to check if α is preferred to β use Greedy algorithm (slight variation of [Wilson et al., IJCAI'15]):
  - Start with empty sequence, i.e., $\pi = ()$.
  - Consider preference statements $\Gamma' = \Gamma \cup \{\alpha < \beta\}$.
  - Repeatedly add new elements $i \in \{1, \ldots, p\}$ to π for which no preference statement in Γ' becomes opposed, i.e., $\nexists(\delta \geq \gamma)$ or $(\delta > \gamma) \in \Gamma$ s. t. $\delta <_\pi \gamma$.
  - Return TRUE is and only if final sequence π satisfies all strict statements.

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- Imprecise Tradeoffs
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Solving MOCOP

Multi-objective AND/OR Branch-and-Bound (MOAOBB) [Marinescu, CP'09]

1. Create primal graph and pseudo-tree.
2. Perform depth-first Branch-and-Bound search on the associated weighted AND/OR search tree.
3. Backtrack if lower bound dominates upper bound for the set of undominated solutions.

MOAOBB

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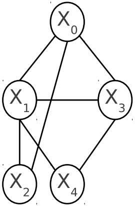
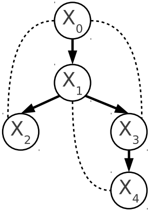
X_0	X_1	X_3	f_2
0	0	0	(1,5)
0	0	1	(4,2)
0	1	0	(0,6)
0	1	1	(2,4)
1	0	0	(6,1)
1	0	1	(5,5)
1	1	0	(6,4)
1	1	1	(5,2)

X_1	X_3	X_4	f_3
0	0	0	(1,4)
0	0	1	(3,2)
0	1	0	(5,1)
0	1	1	(4,3)
1	0	0	(1,9)
1	0	1	(3,2)
1	1	0	(5,6)
1	1	1	(4,8)

(a) utility functions



(b) primal graph



(c) pseudo tree

MOAOBB

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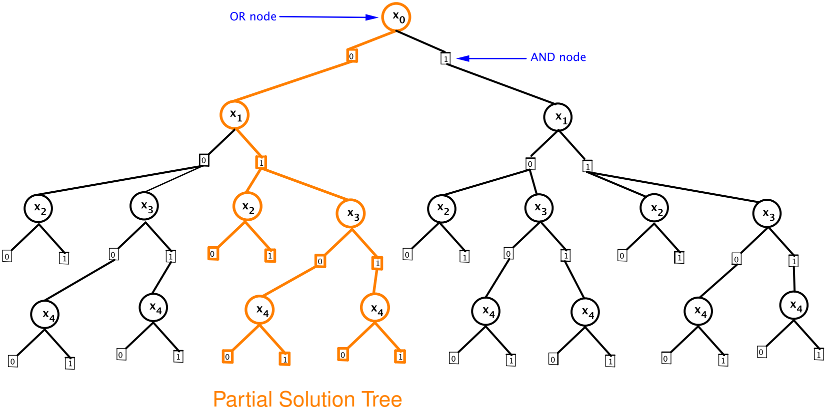

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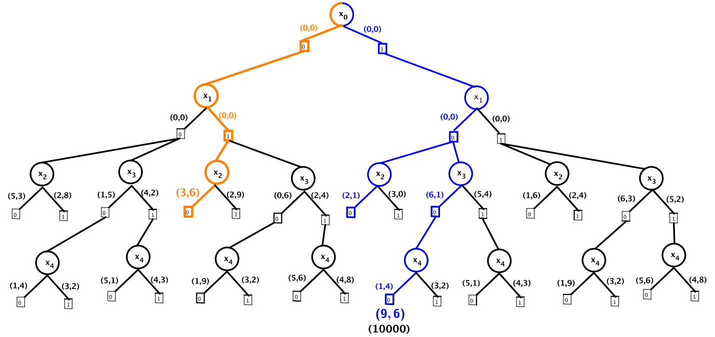

Solution Tree

MOAOBB

1. Create primal graph and pseudo-tree.
2. Perform depth-first Branch-and-Bound search on the associated weighted AND/OR search tree.
3. Backtrack if lower bound dominates upper bound for the set of undominated solutions.

Upper Bound $v(n)$ is maintained for every subproblem descending from node n. Depending on its child nodes and edge weights. Updated in every step.

Lower Bound is given for every subproblem descending from node n by heuristic $h(n)$. E.g., Static or Dynamic Mini-Bucket Heuristic.

Dominance Test $A \succcurlyeq B \Leftrightarrow$ for all $\beta \in B$ exists $\alpha \in A$ such that $\alpha \geq \beta$. Depends on order relation!

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Problem Instances

- Problem types:
  - Vertex covering: 2,3 and 5 objectives
  - Random networks: 2,3 and 5 objectives
  - Auctions: 3 objectives (price, failure, quality)
- Lex/WA-Consistent random trade-offs:
  - Of the form $+, +, -$, e.g., $(1, 0, 0, 2) \geq (0, 0, 1, 0)$

Comparison between Pareto, Weighted and Lex Models

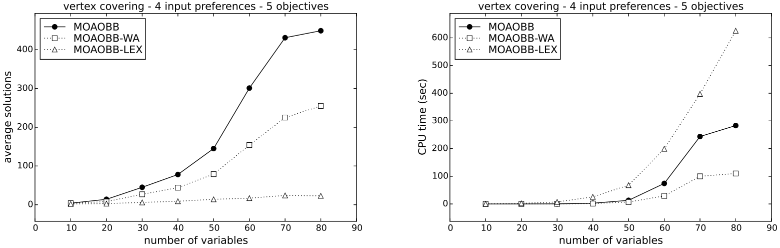

Figure: Average number of solutions (left) and CPU time in seconds for computing (right). Time limit 20 minutes.

Conclusion

Experimental results show, that

- Lex-order yields less solutions than WA- (and Pareto-) order...
- ...for the cost of a higher running time....

... which is consistent with the theoretical results.

Questions?

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Impact of tradeoffs

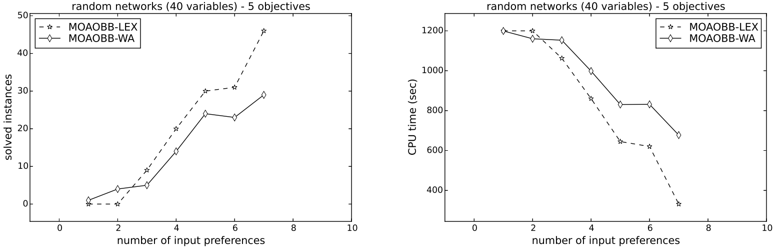

Figure: Number of problems solved (left) and CPU time in seconds (right) for random networks problem instances with 40 variables and 5 objectives. Time limit 20 minutes.

MOAOBB

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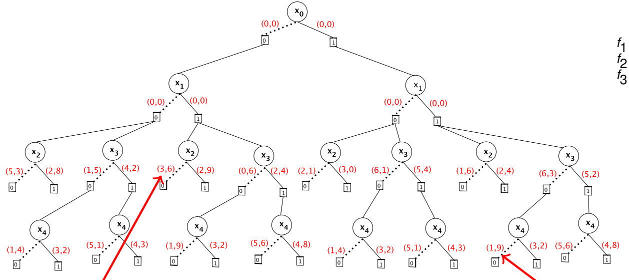
3. Backtrack if lower bound dominates upper bound for the set of undominated solutions.

$f_1 : \{X_0, X_1, X_2\}$
$f_2 : \{X_0, X_1, X_3\}$
$f_3 : \{X_1, X_3, X_4\}$



Weighted AND/OR search tree

$f_1(0_{X_0}, 1_{X_1}, 0_{X_2}) = (3, 6)$

$f_3(1_{X_1}, 0_{X_3}, 0_{X_4}) = (1, 9)$