# On the likelihood of randomly perturbed max-solutions

Tamir Hazan
Technion

# Inference in machine learning

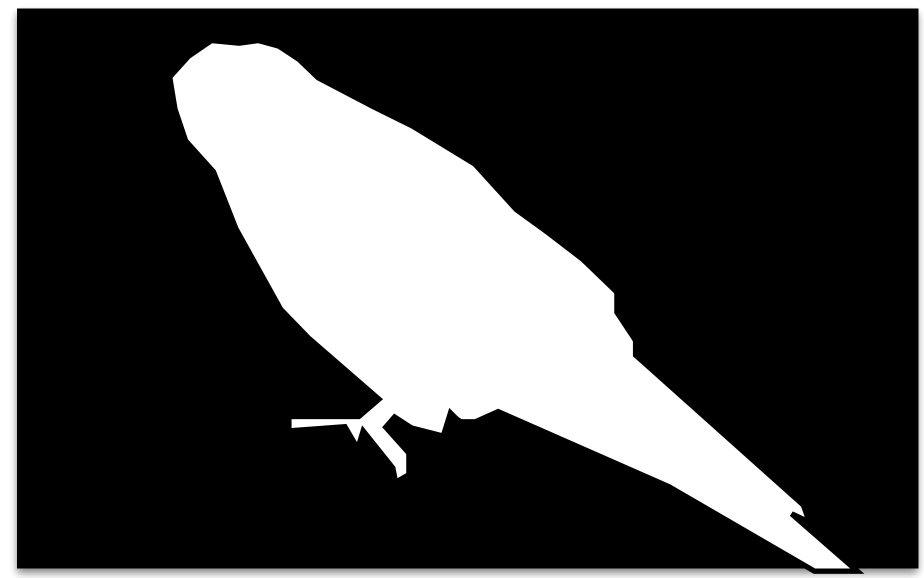
- 20 years ago: does an image contain a person?

# Inference in machine learning

- 10 years ago: which object is in the image?

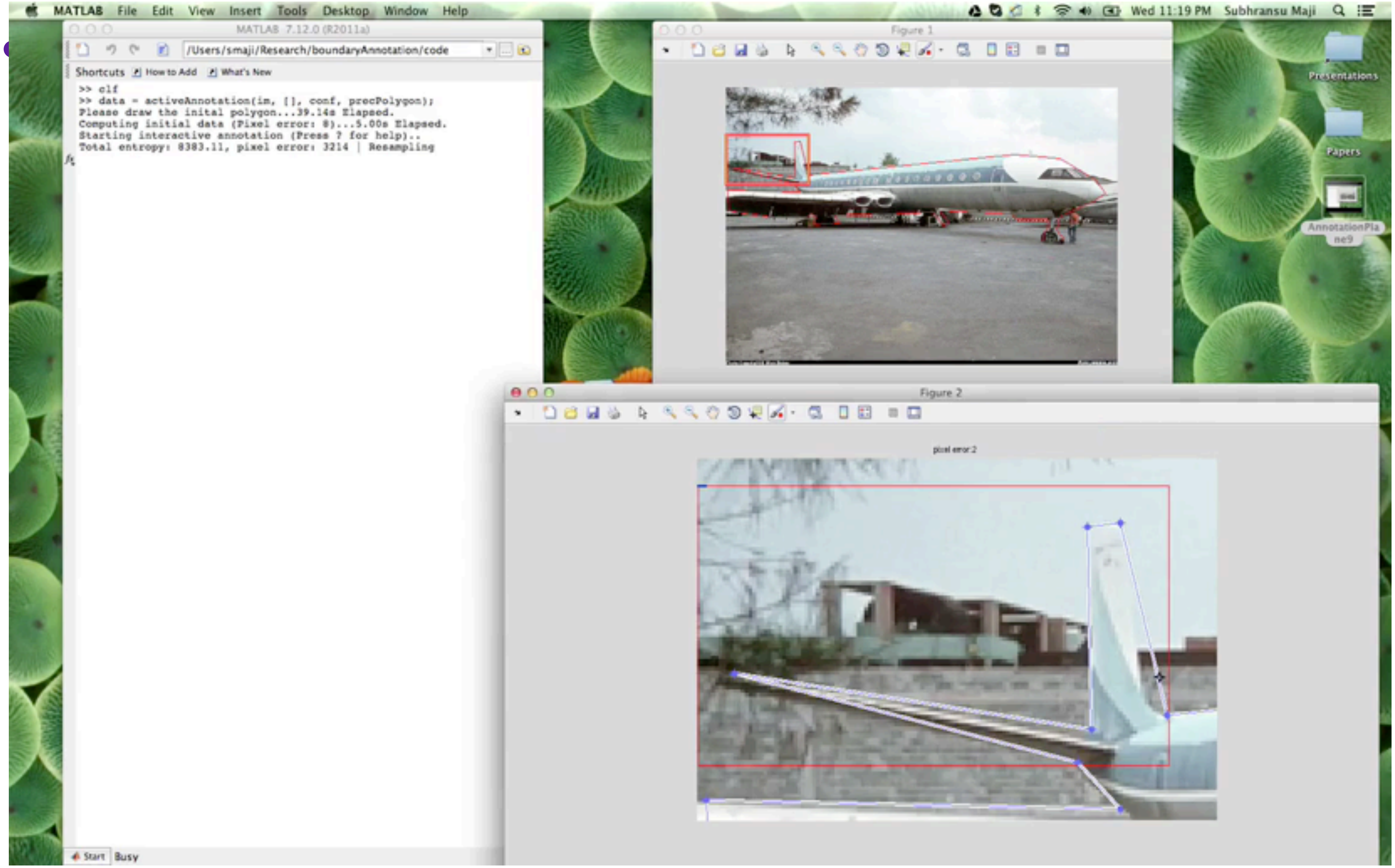
# Inference in machine learning

- Today's challenge:  exponentially many options



- For each pixel: decide if it is foreground or background.
- The space of possible structures is exponential

# Inference in machine learning



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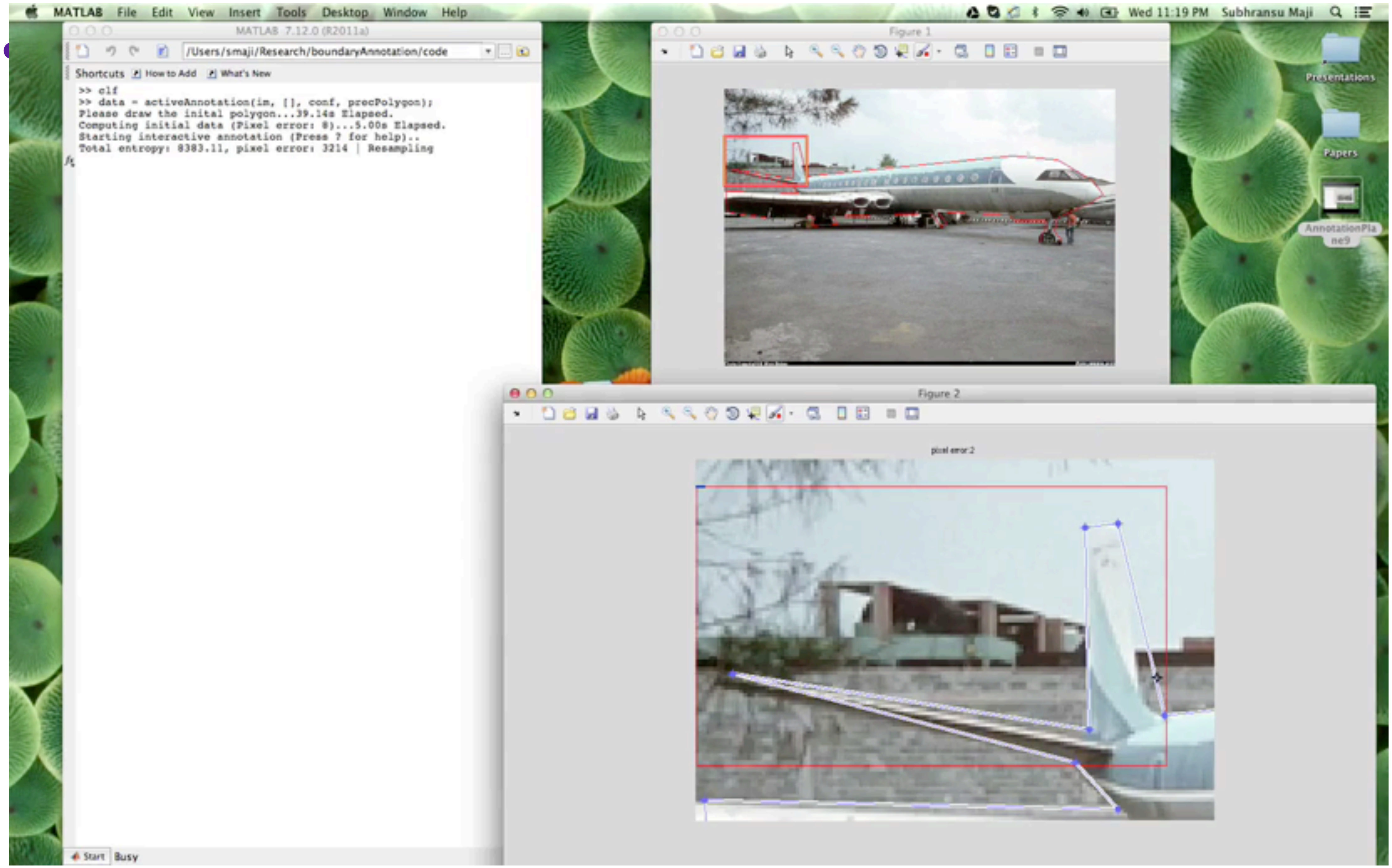

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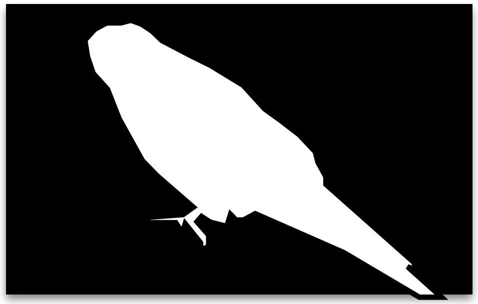

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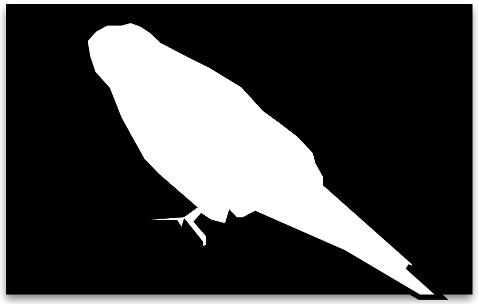
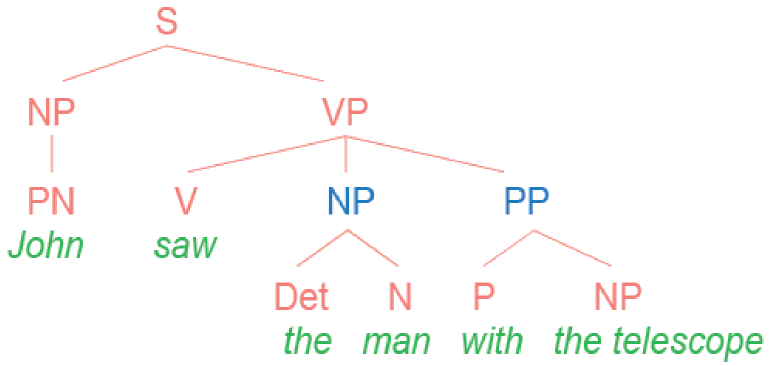


  - Natural language processing



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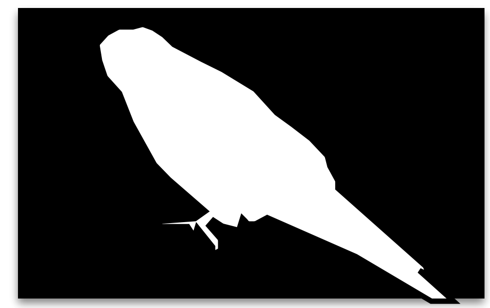

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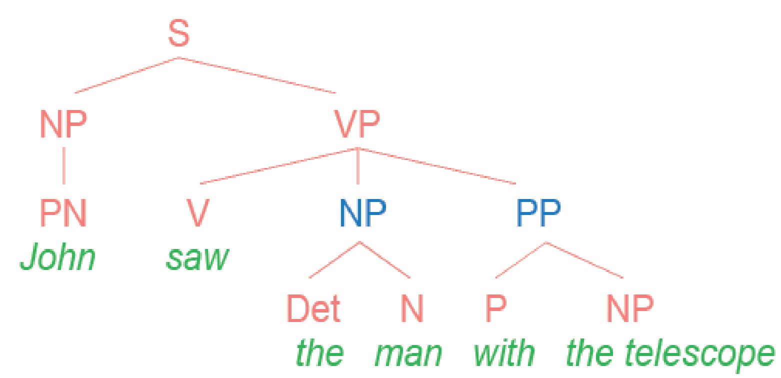
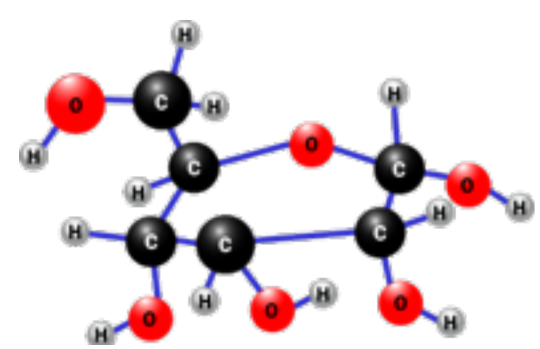



  - Computational biology



  - and more..

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- Connections and Alternatives to Gibbs distribution:
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  - the modeling power of perturb-max models
- Learning with perturb-max models
  - log-likelihood learning
  - interactive learning using new entropy bounds
  - online learning
  - loss minimization and PAC-Bayesian bounds

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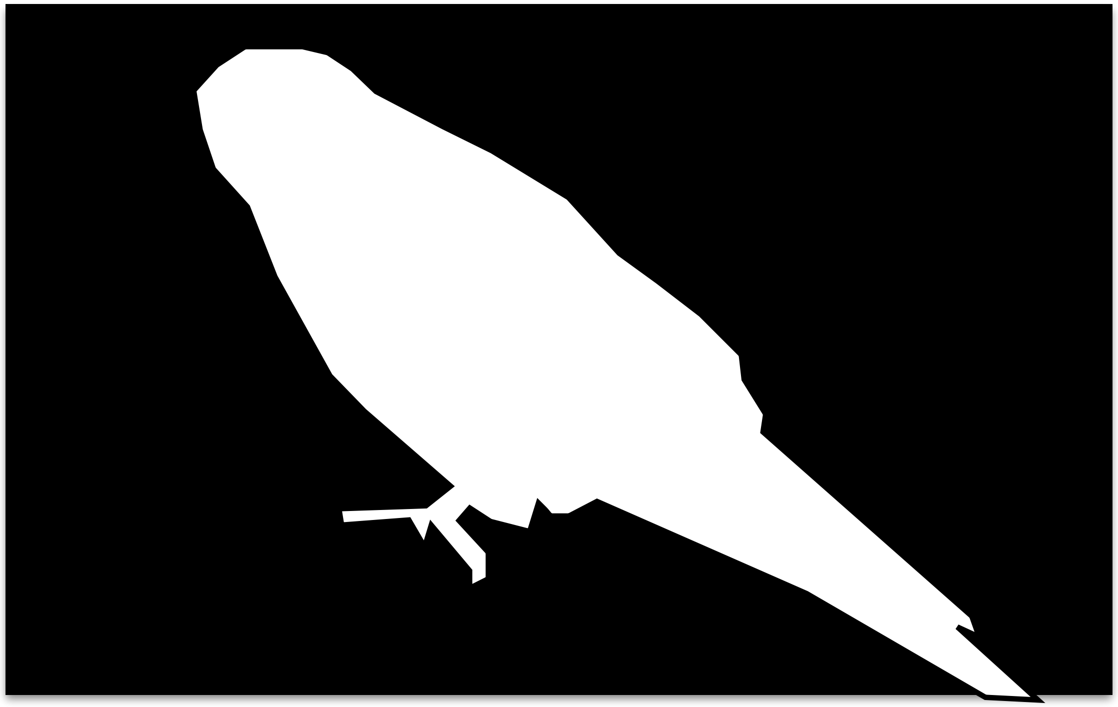

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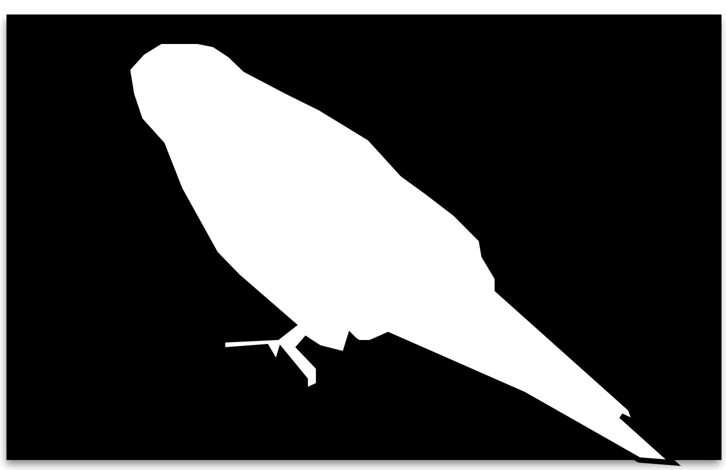

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$$w_{i,j} \geq 0$$

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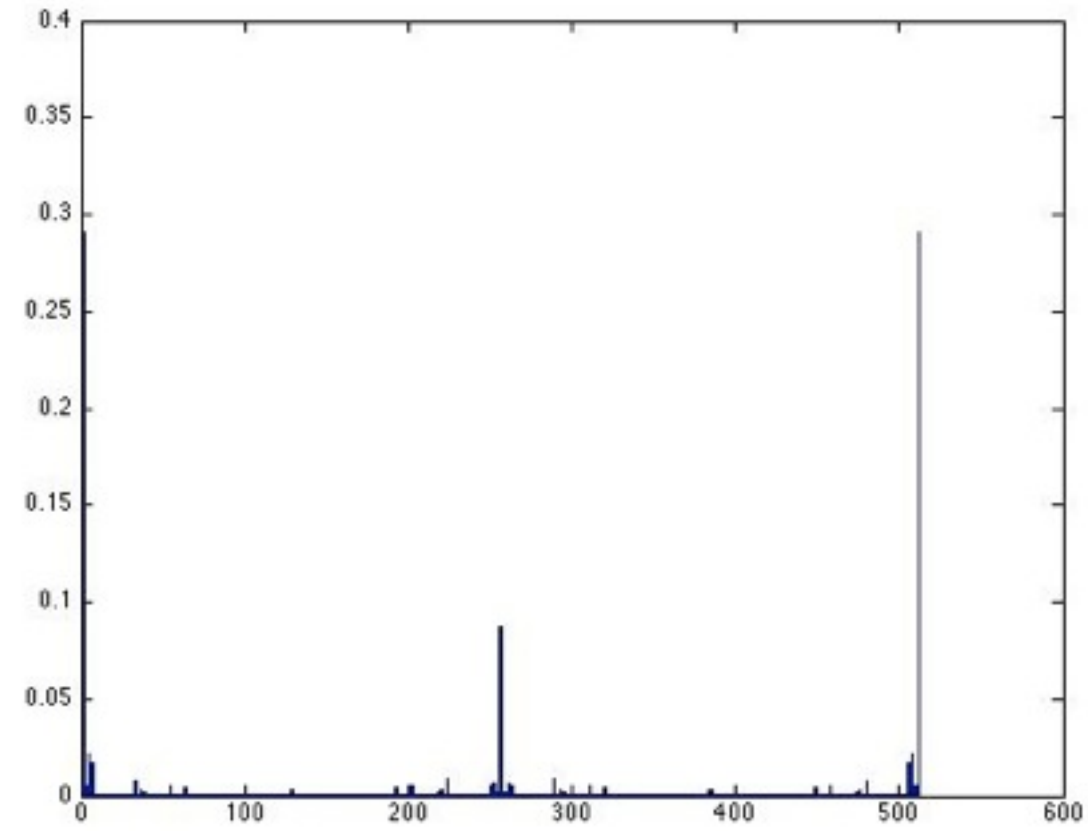

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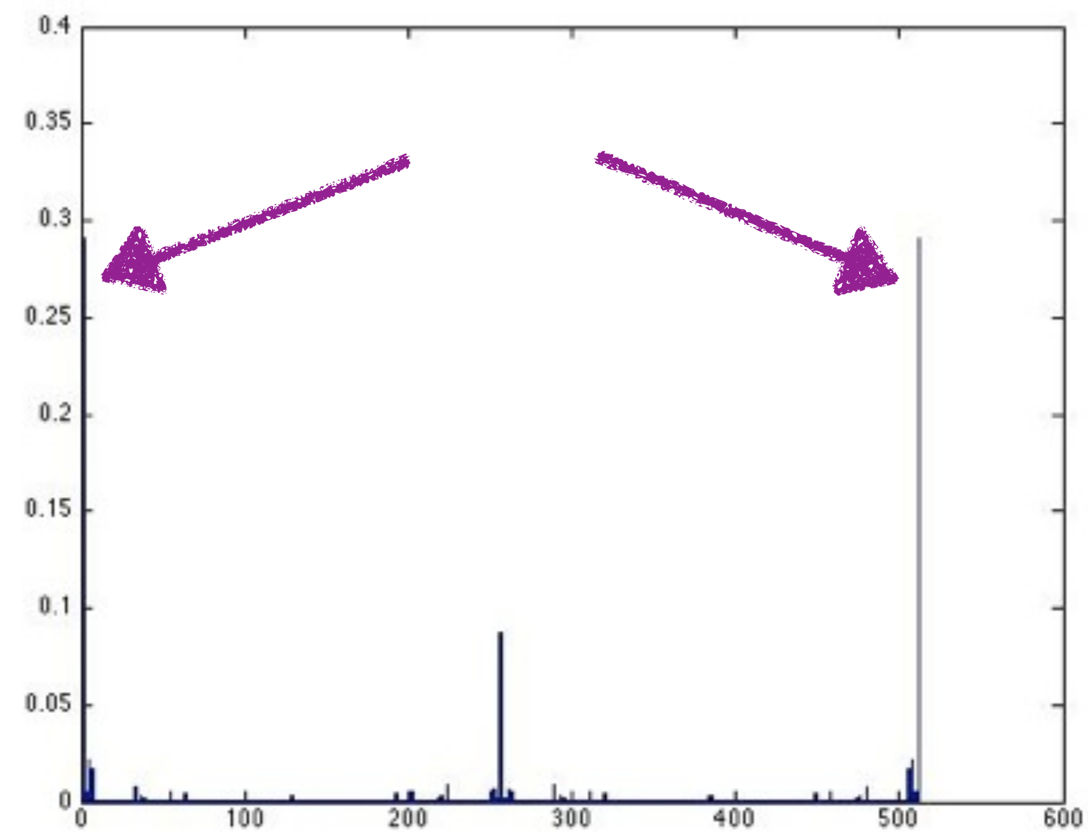


- Nicely behaved distribution that is centered around the (1,…,1) or (0,…,0)

# Sampling likely structures

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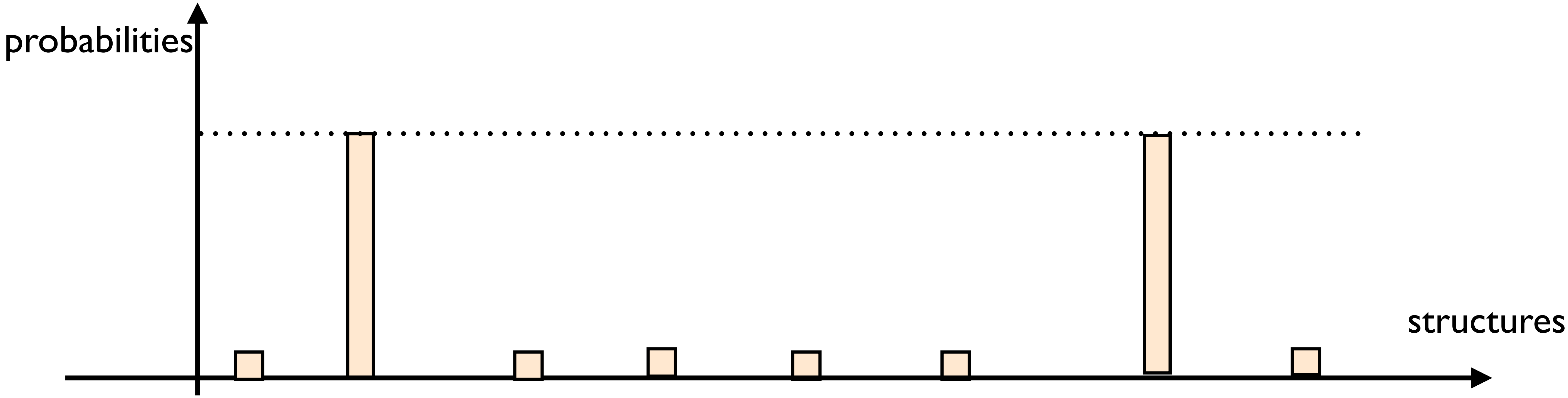

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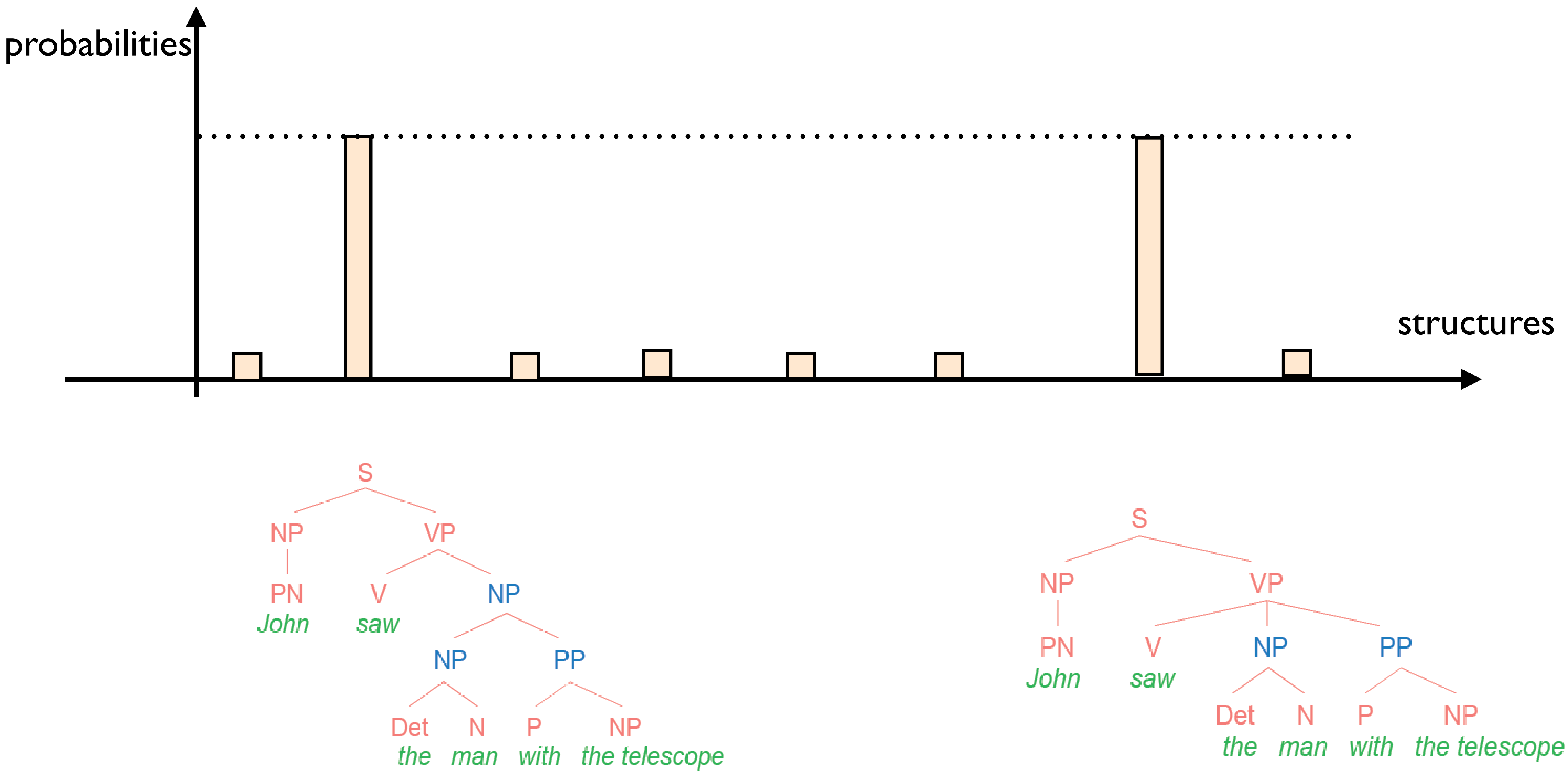

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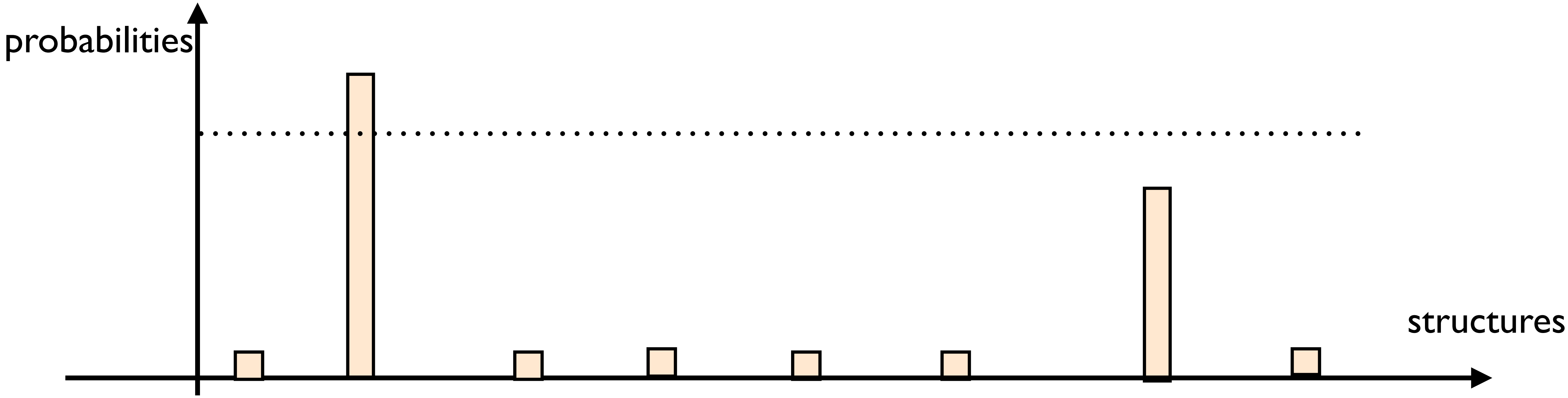

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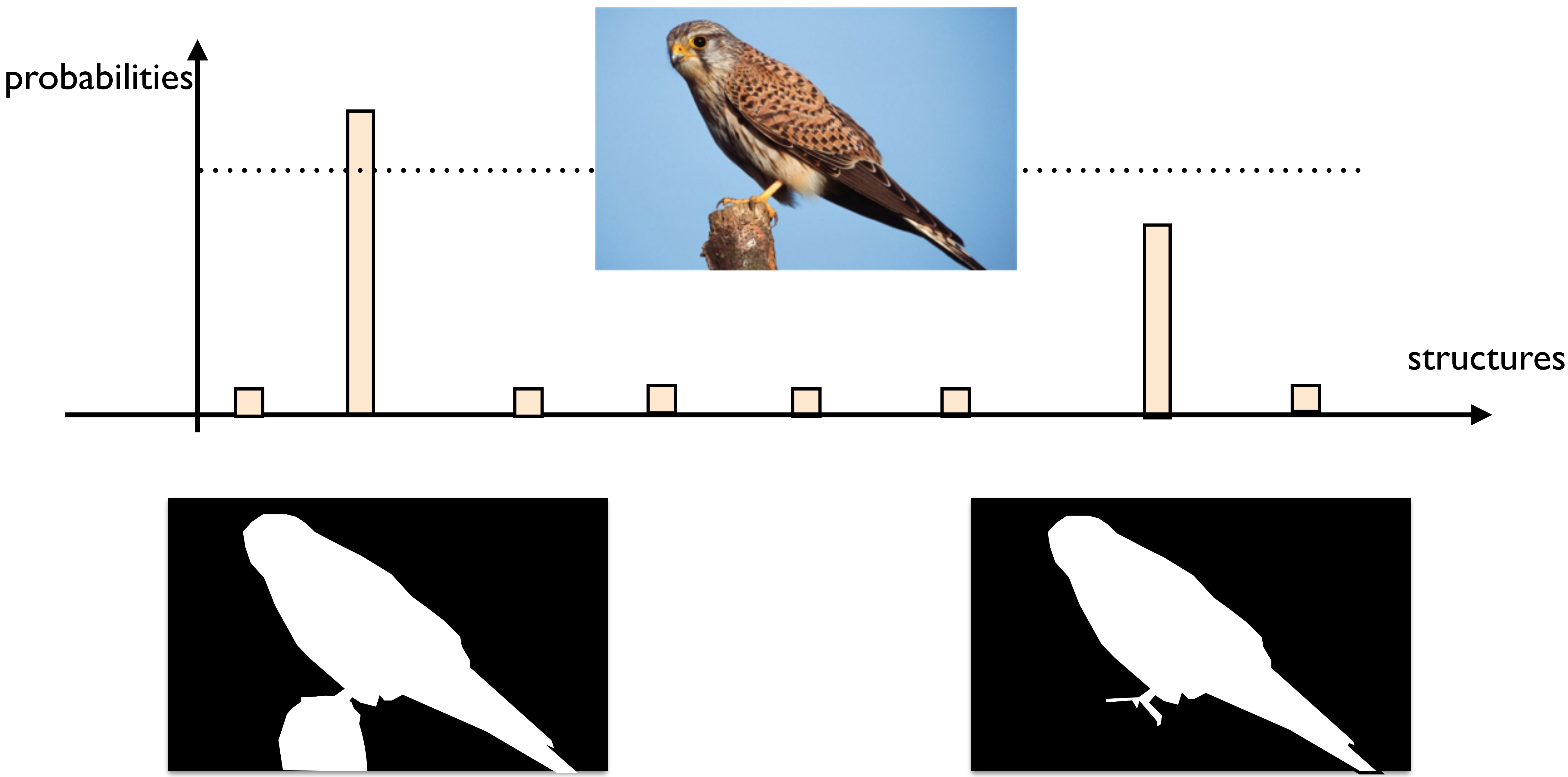

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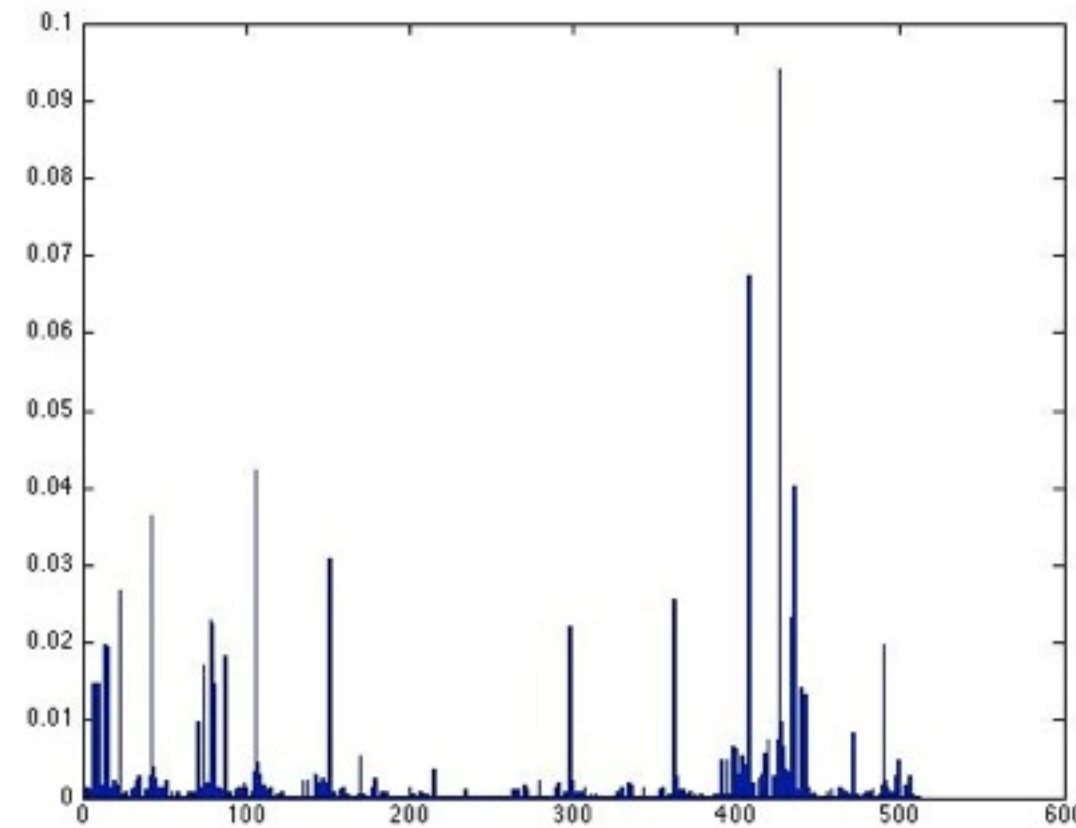

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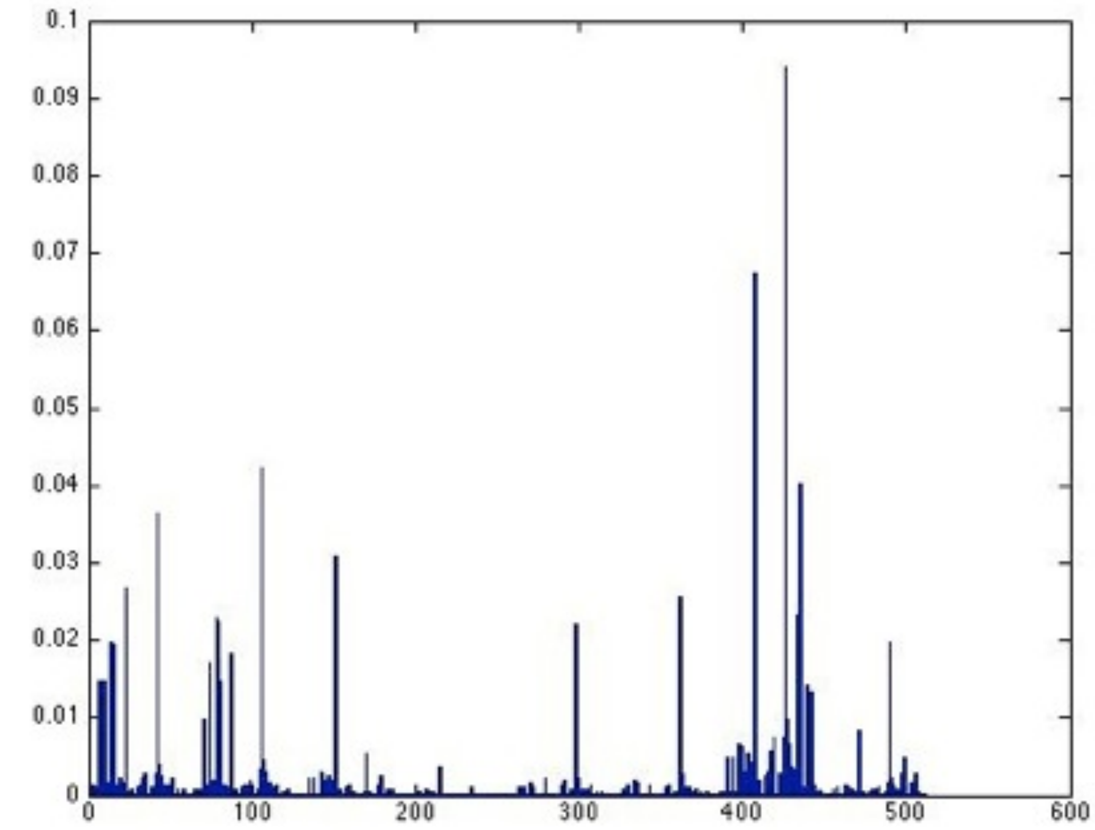

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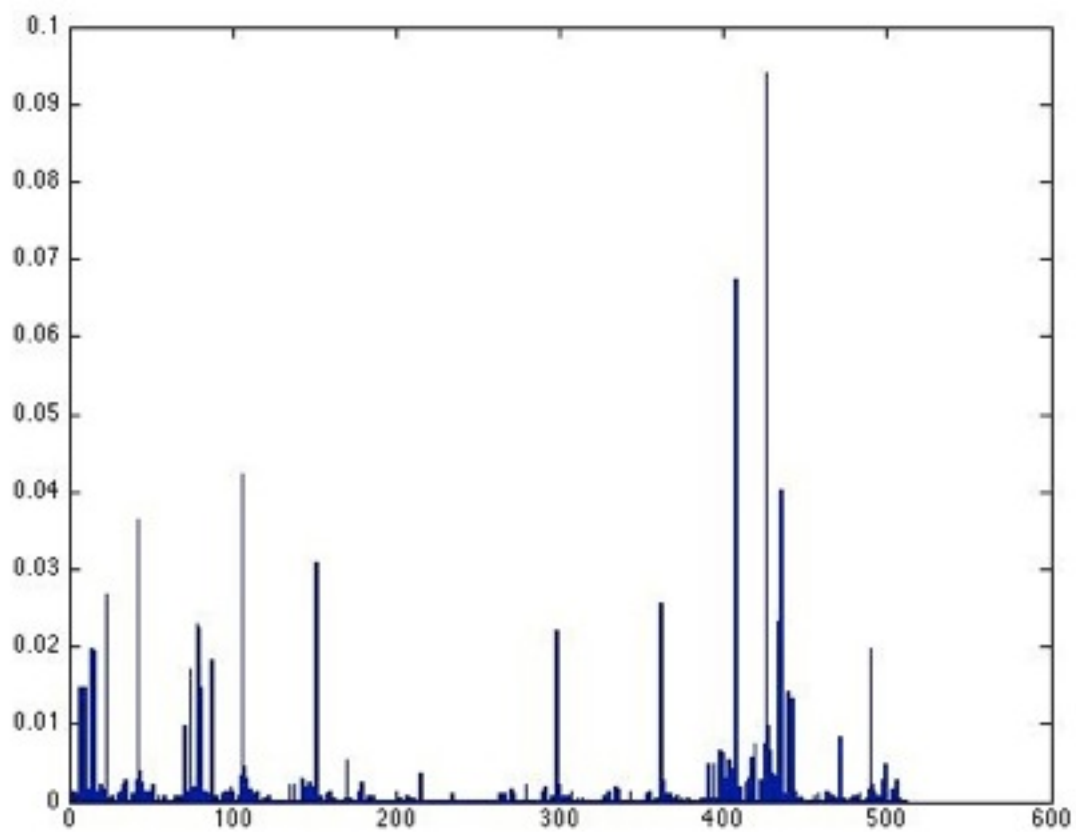


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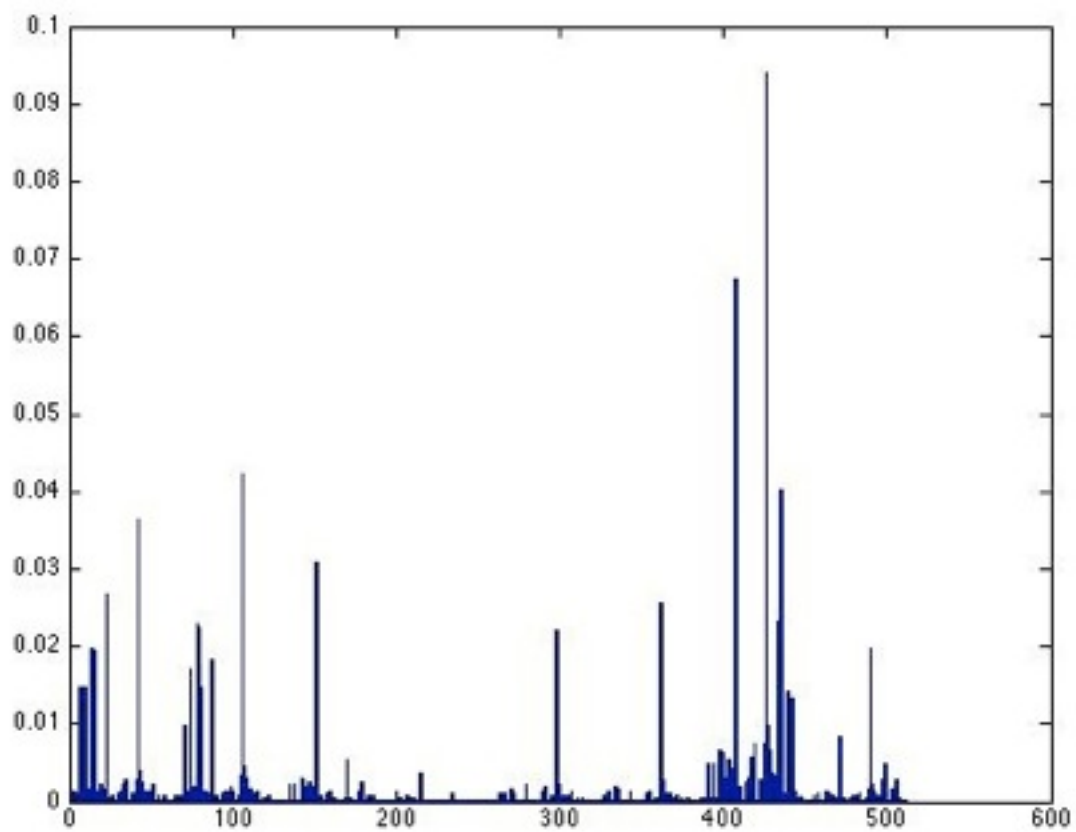


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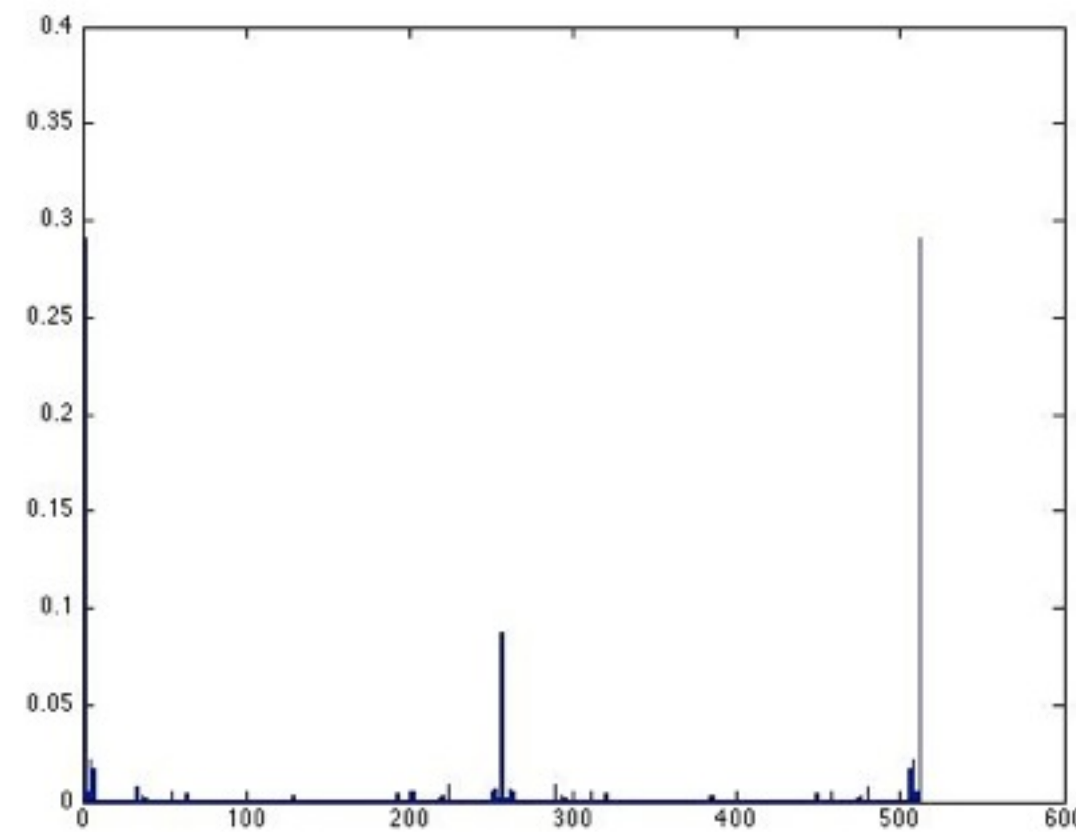


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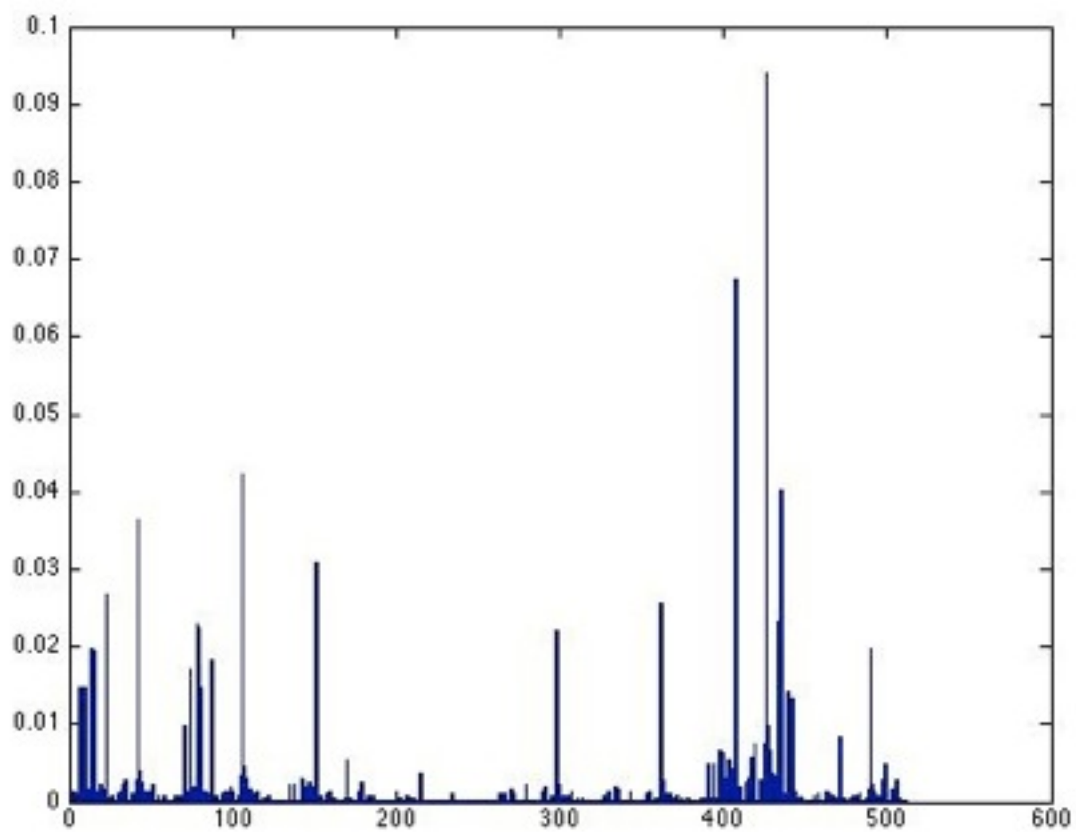


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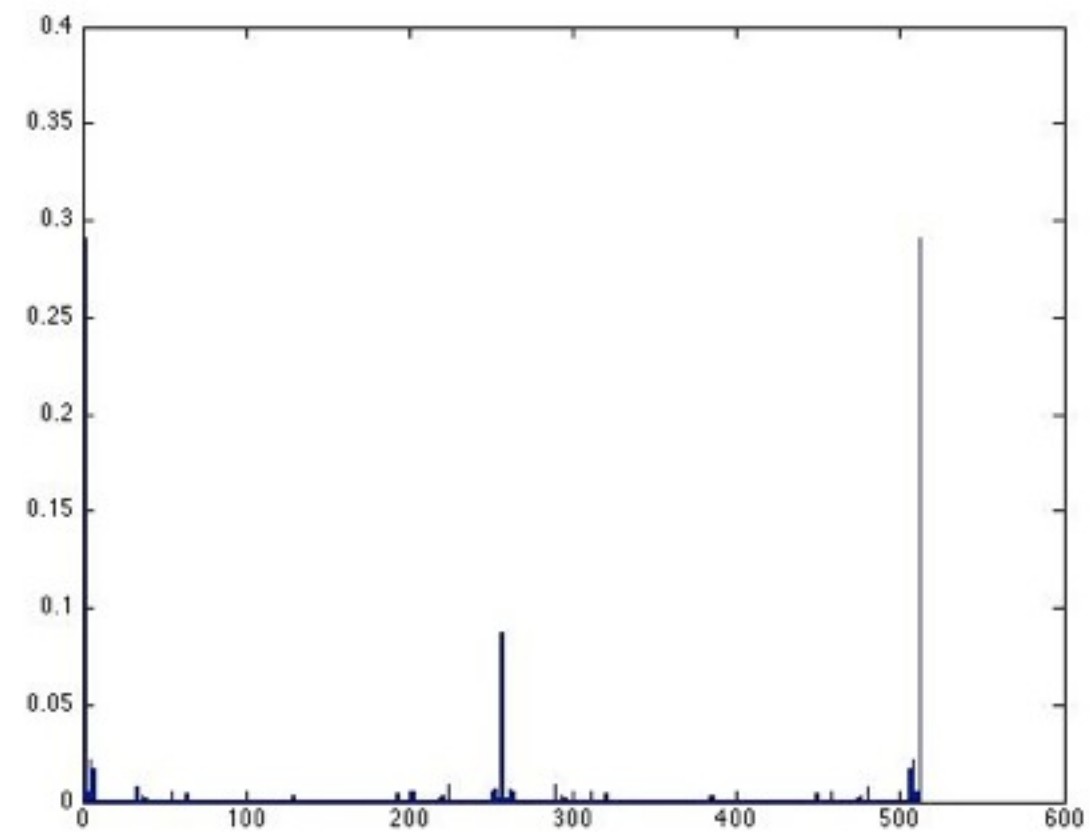


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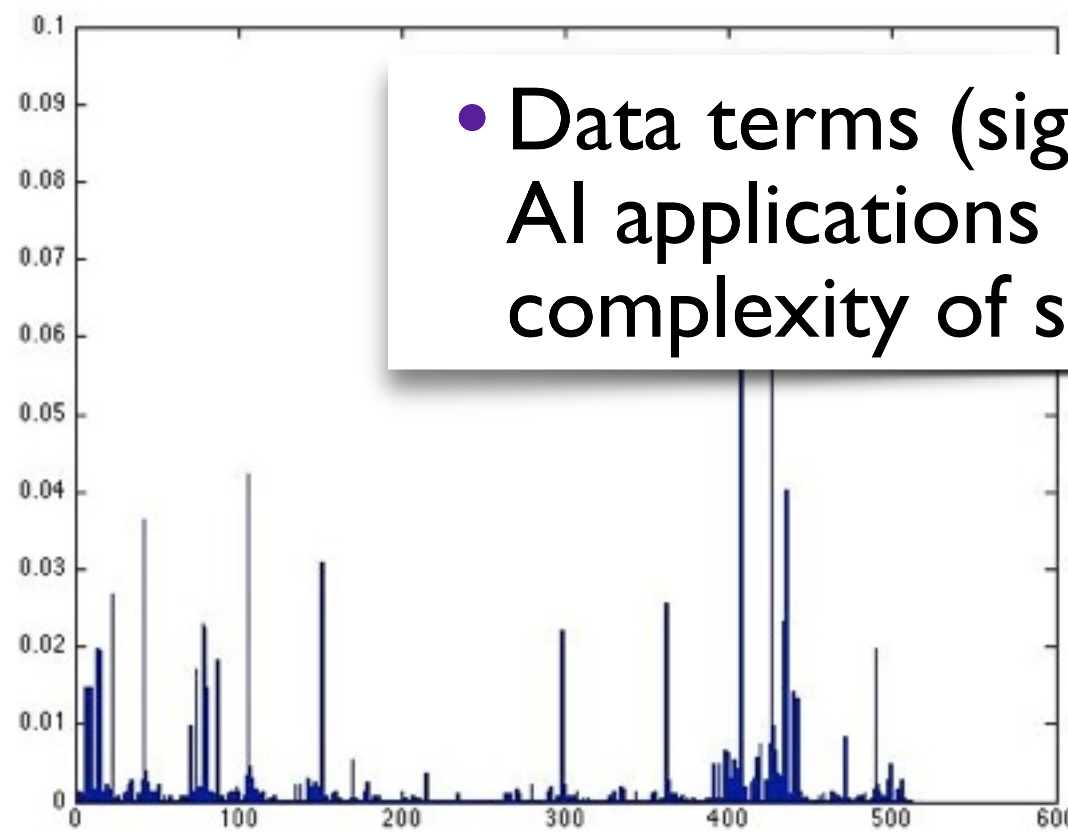


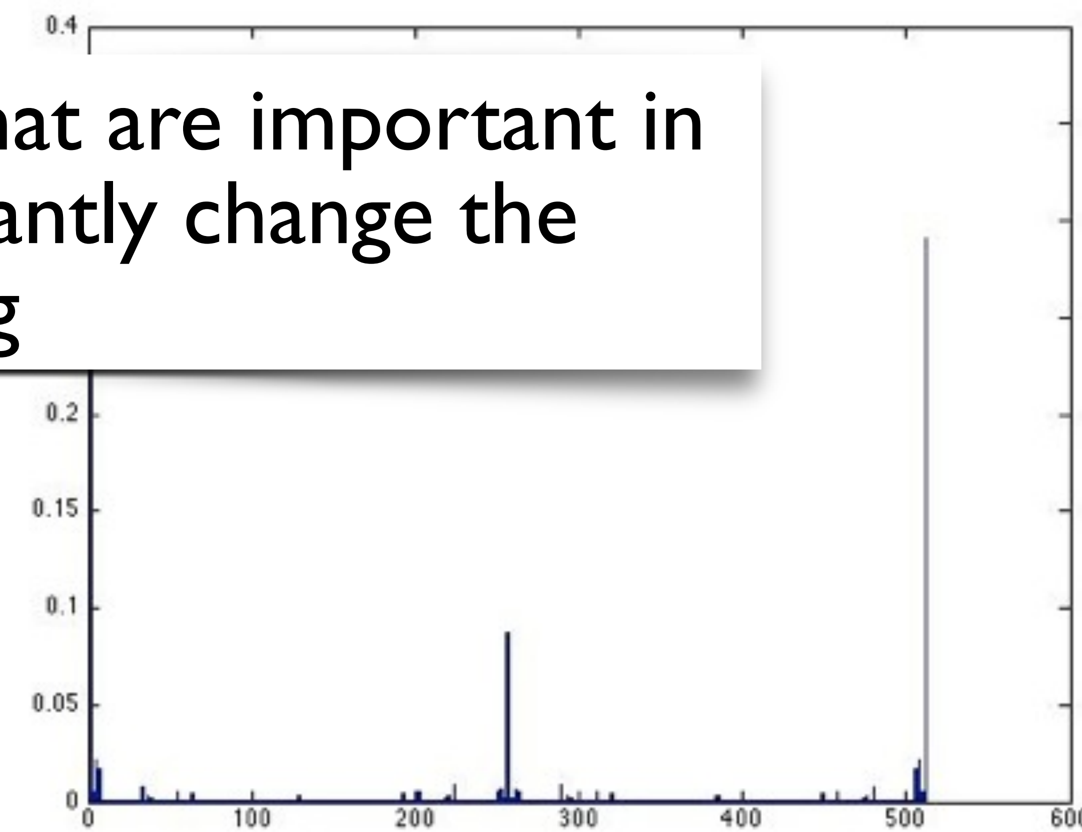


- Data terms (signals) that are important in AI applications significantly change the complexity of sampling

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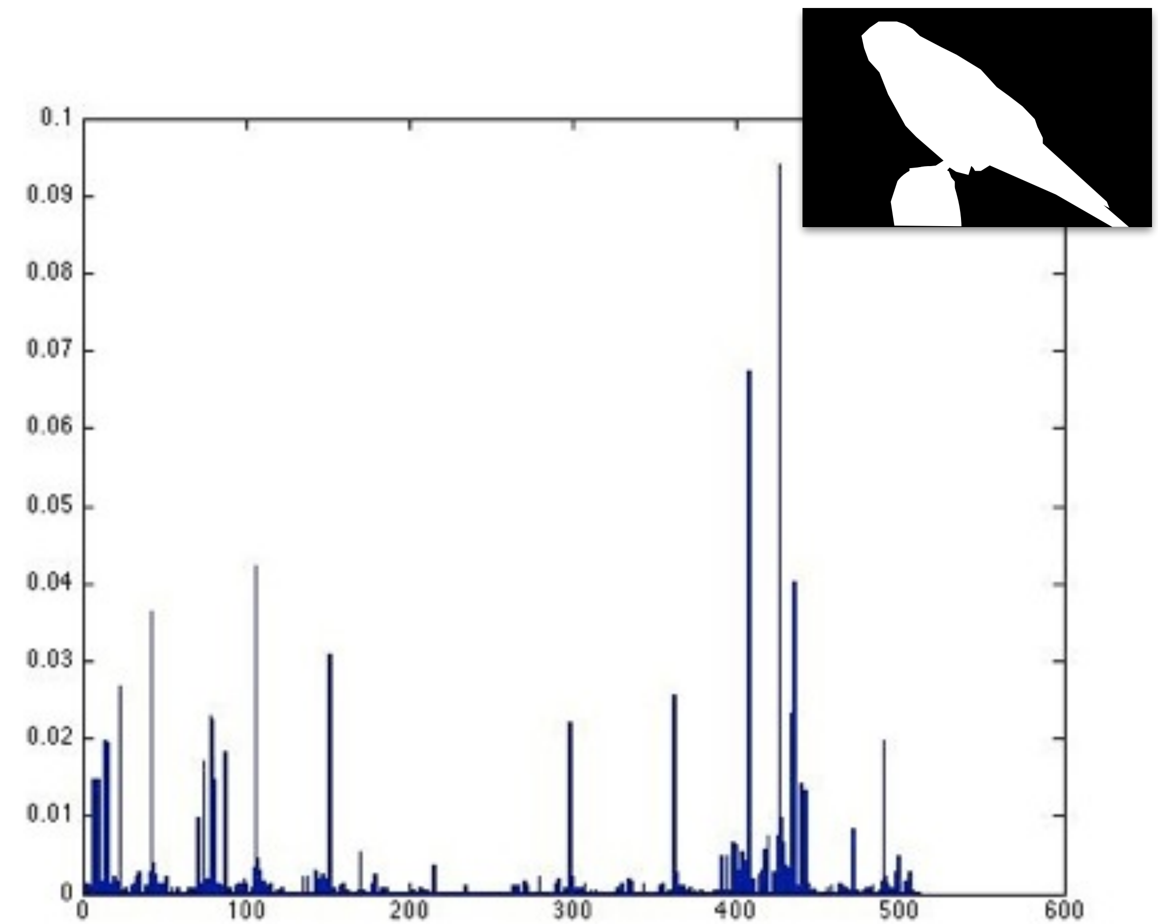

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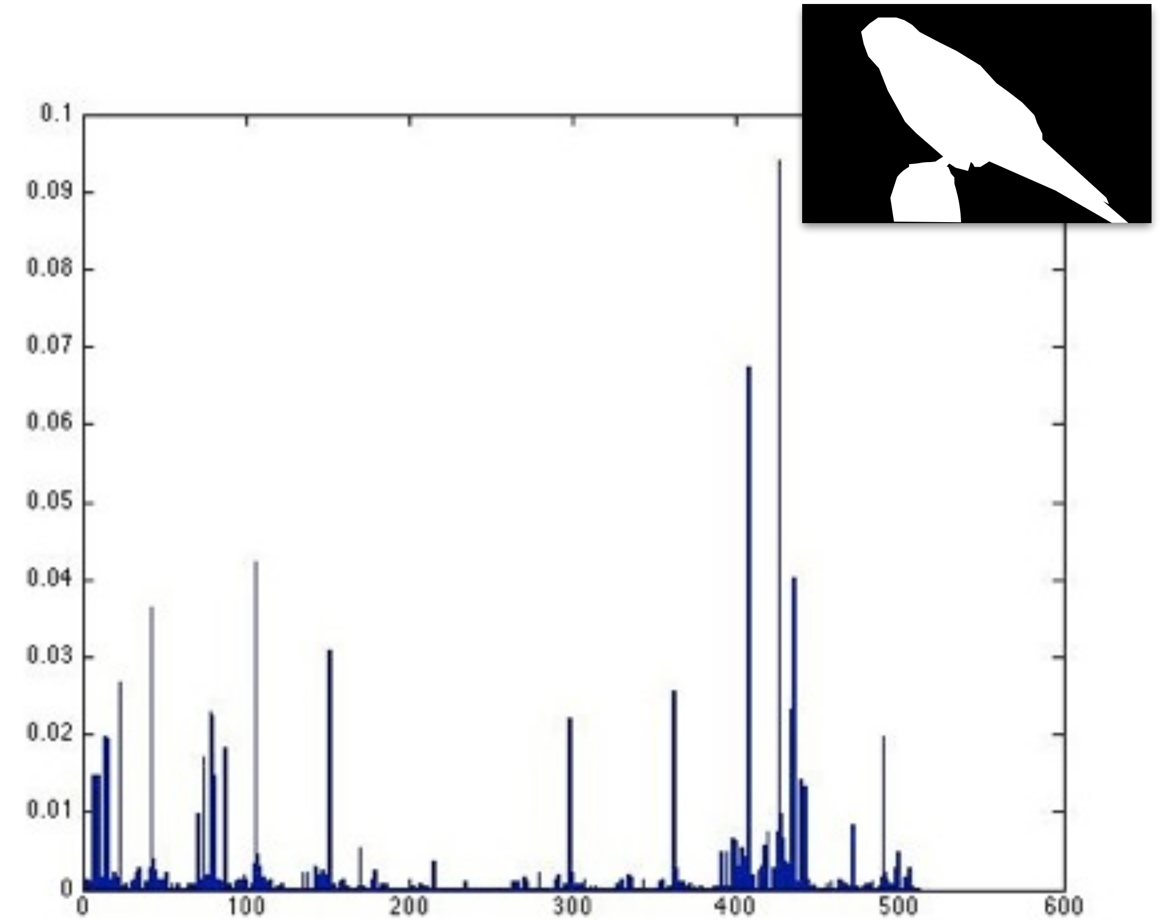

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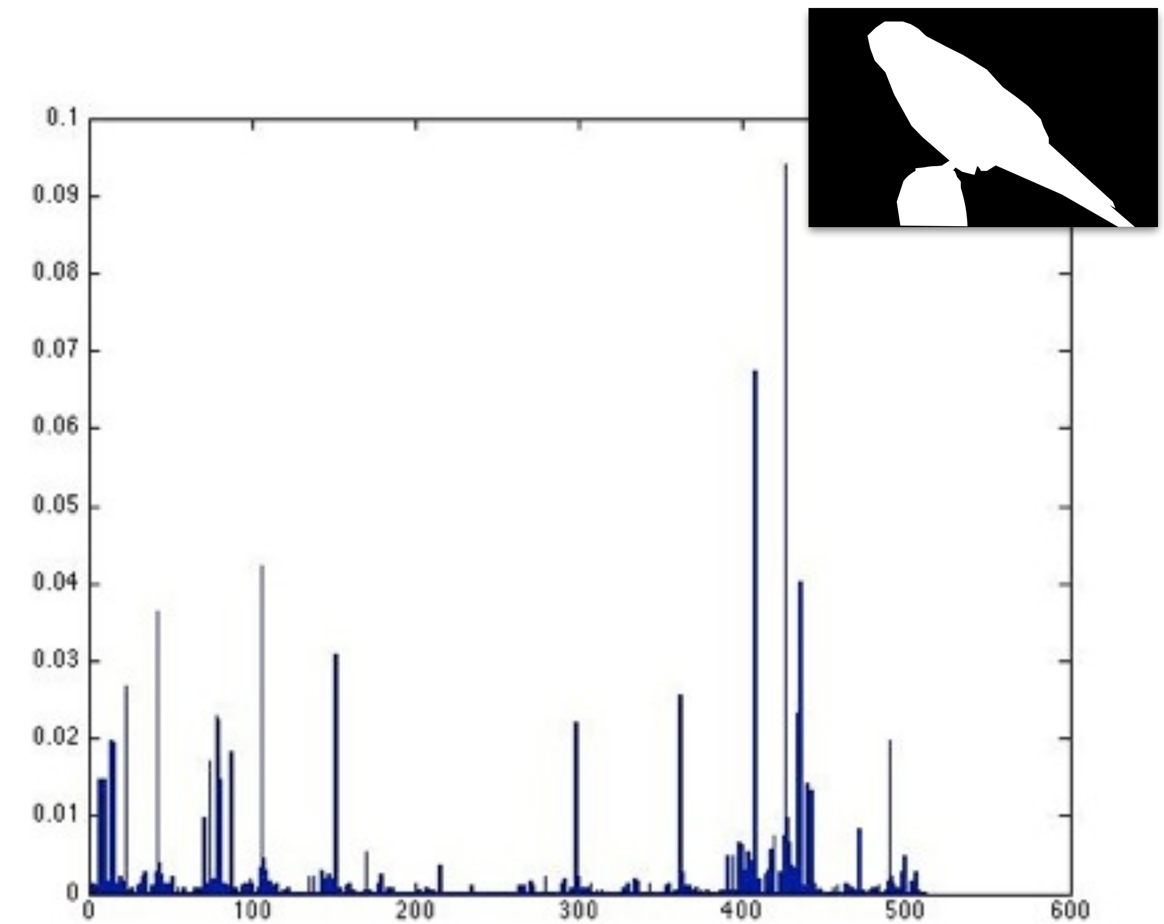

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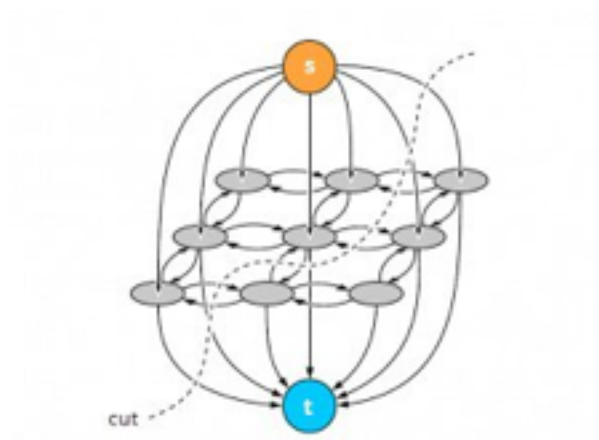


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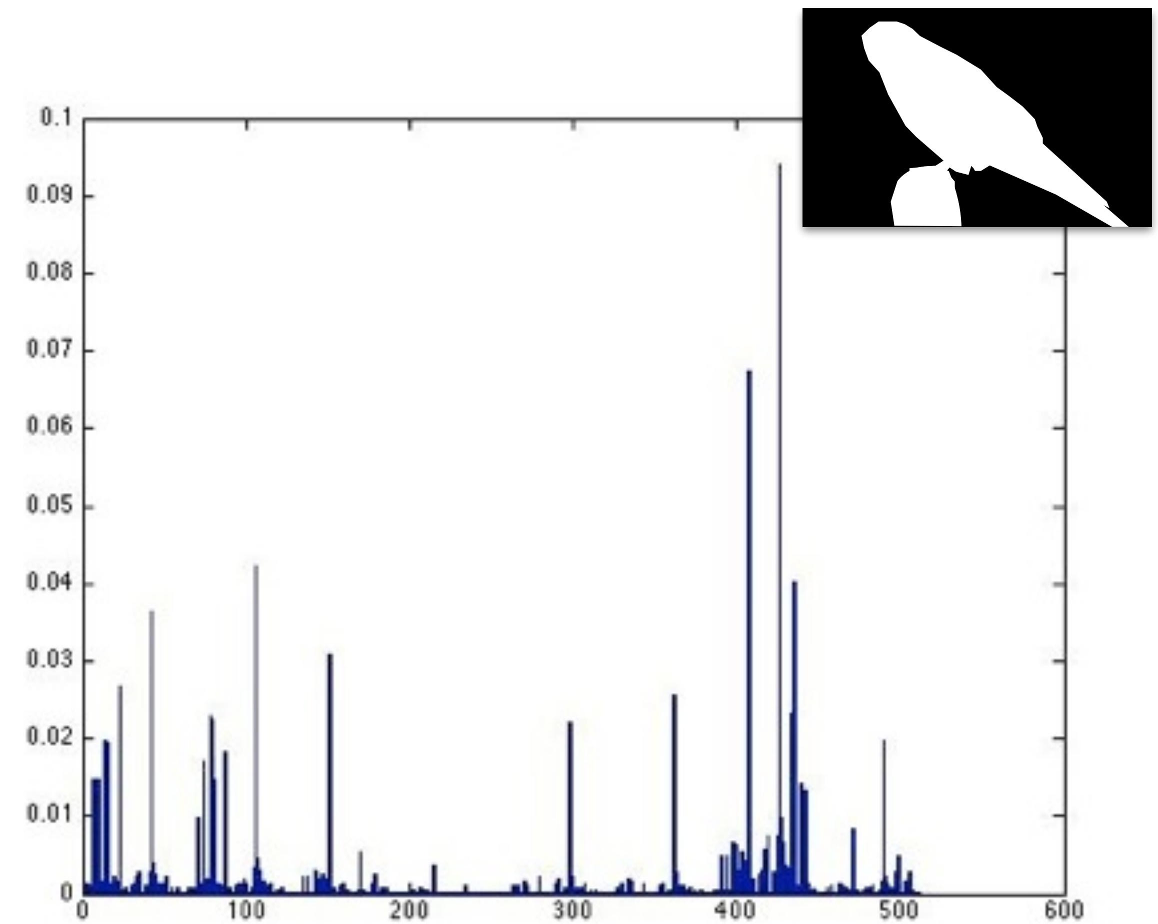

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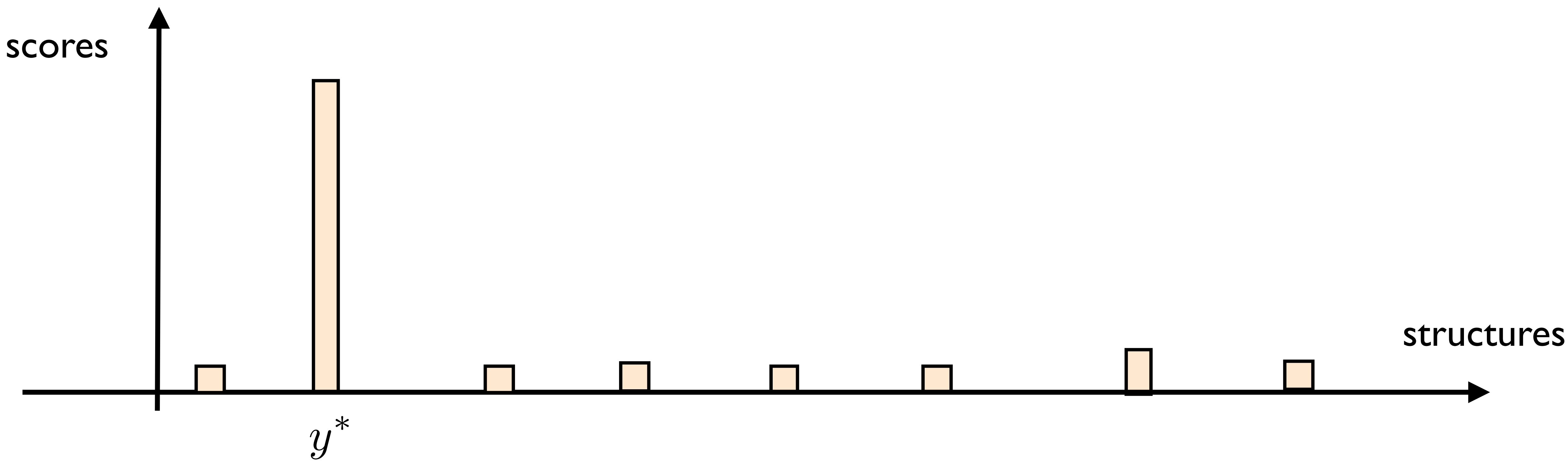
  - Linear programming relaxations (Schlesinger 76, Wainwright 05, Kolmogorov 06, Komodakis 07, Werner 07, Sontag 08, Hazan 10, Kappes 13, Savchynskyy13, Tarlow 13)
  - CKY for parsing
  - Many others…

# The challenge

Sampling from the likely high dimensional structures (with millions of variables, e.g., image segmentation with 12 million pixels) as efficient as optimizing

# Most likely structure

- Selecting the maximizing structure is appropriate when one structure (e.g., segmentation / parse) dominates others



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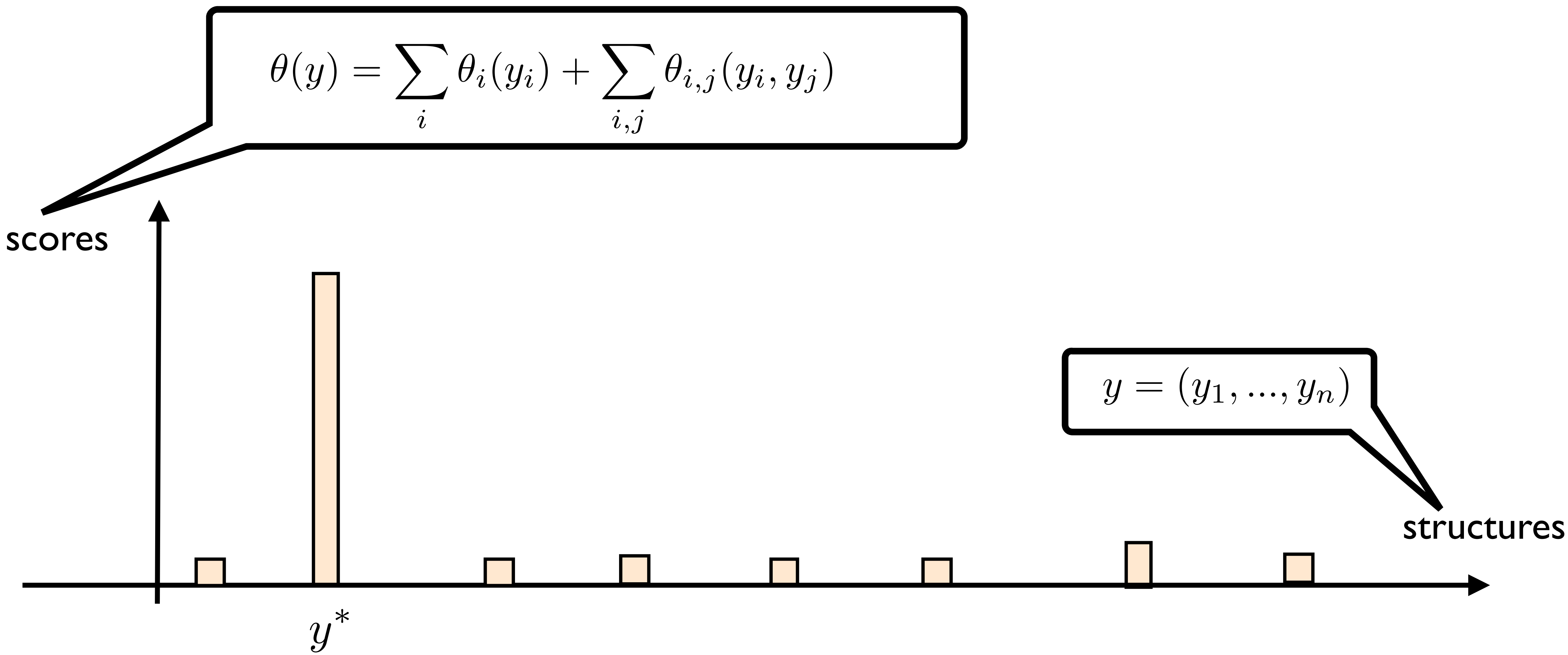

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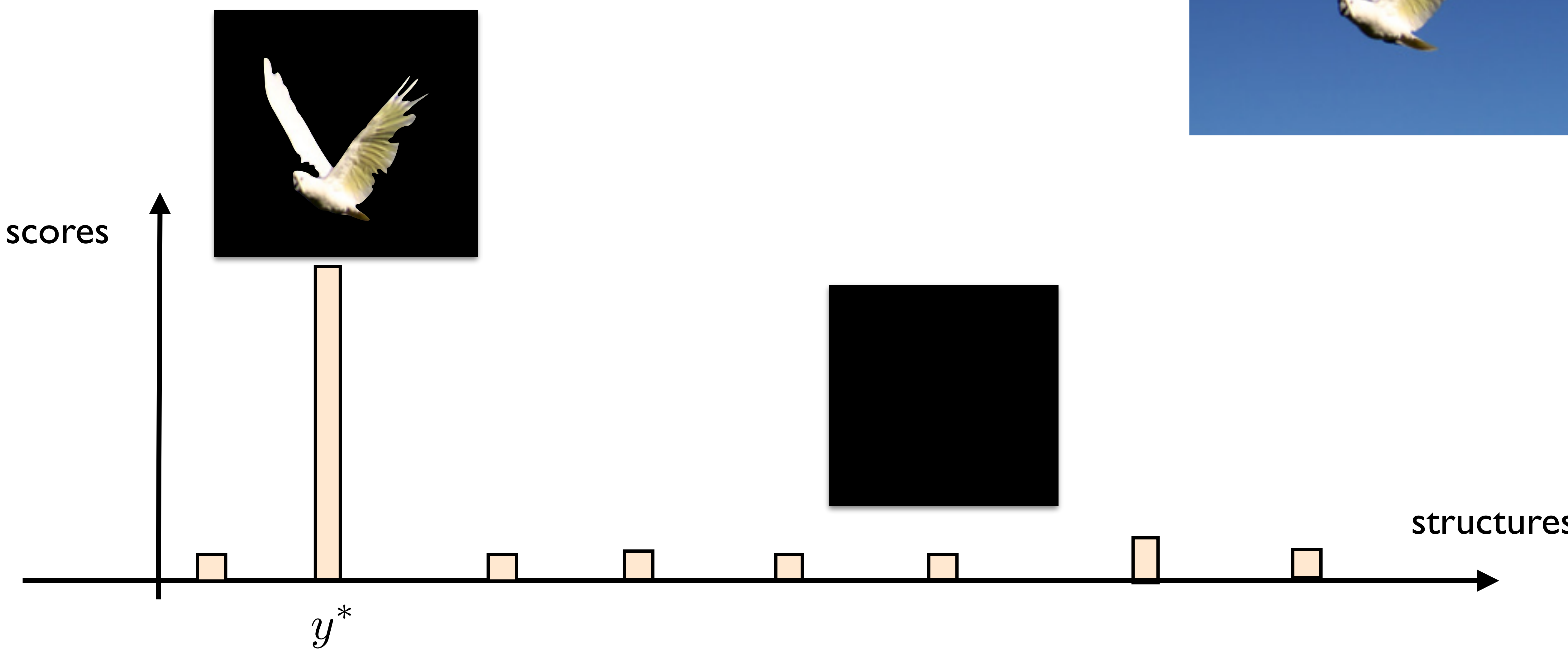

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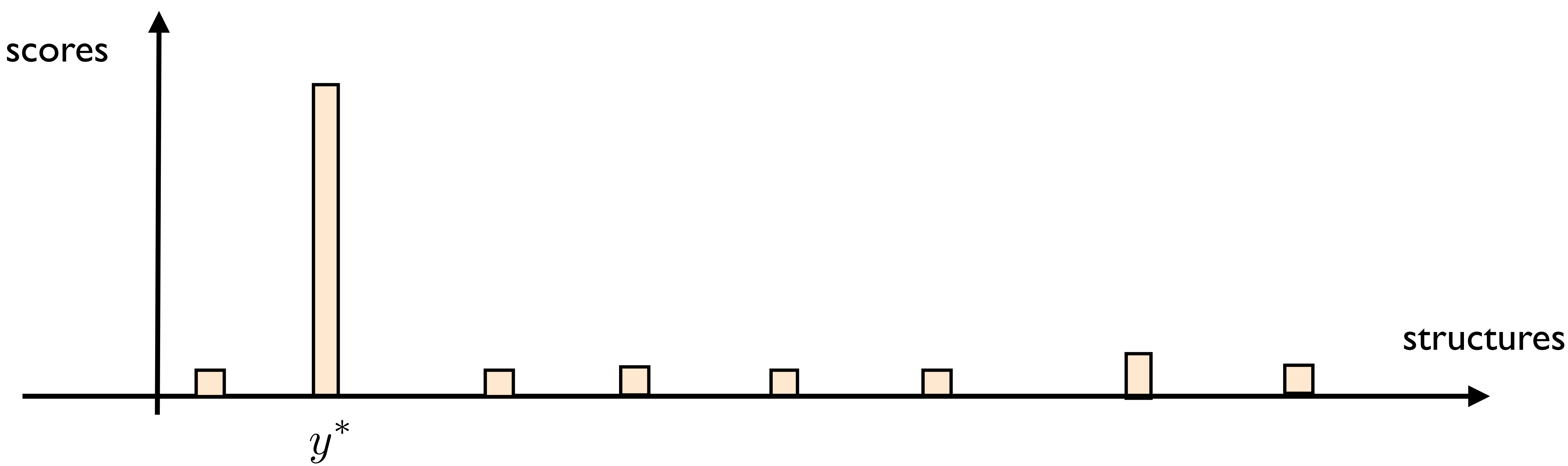

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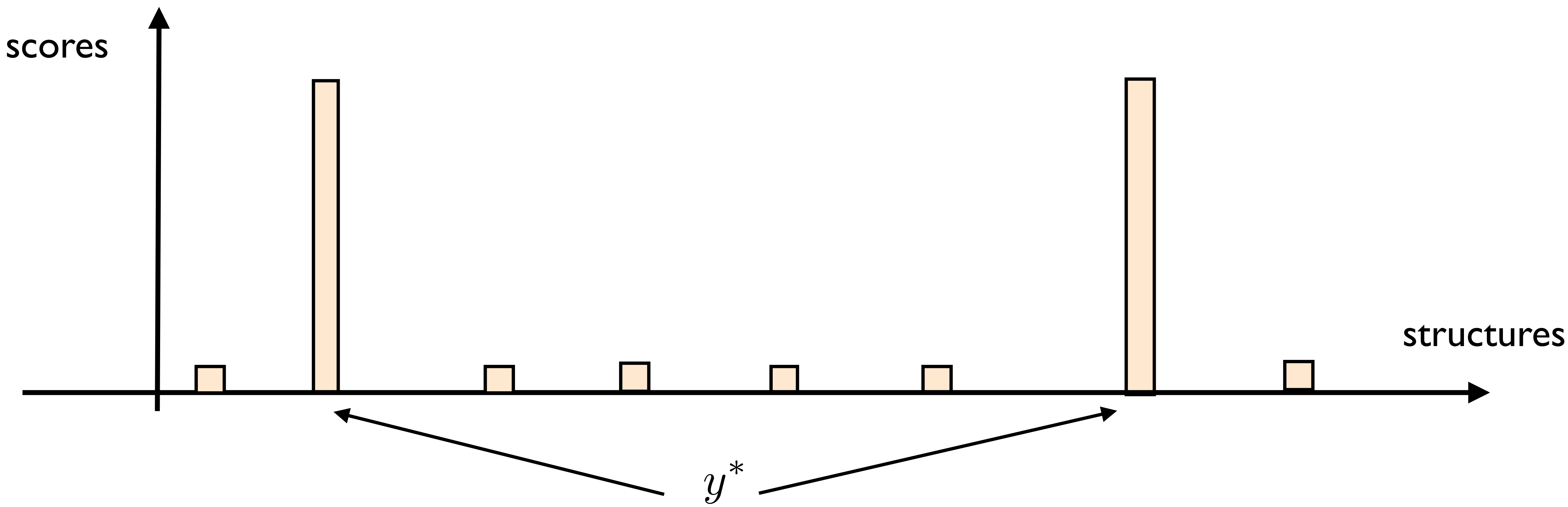

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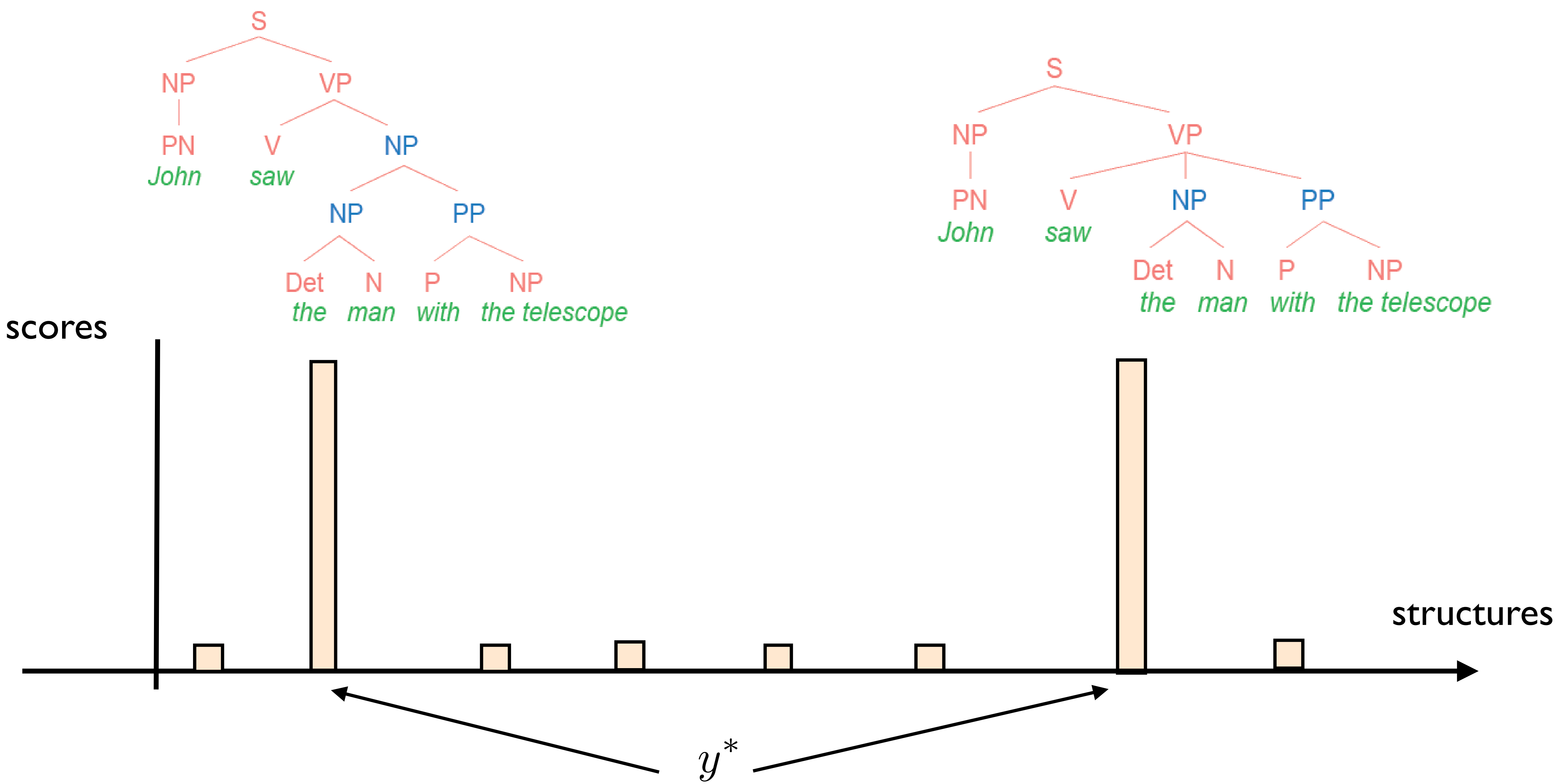

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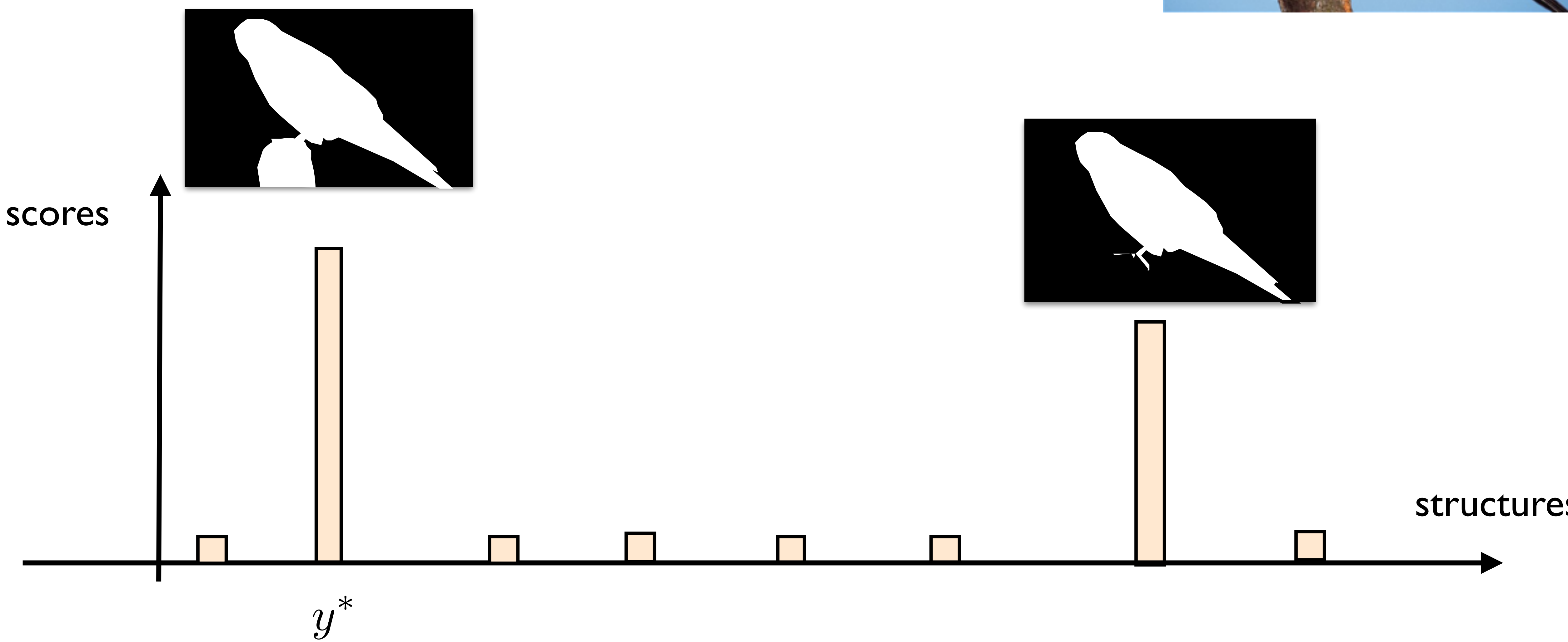

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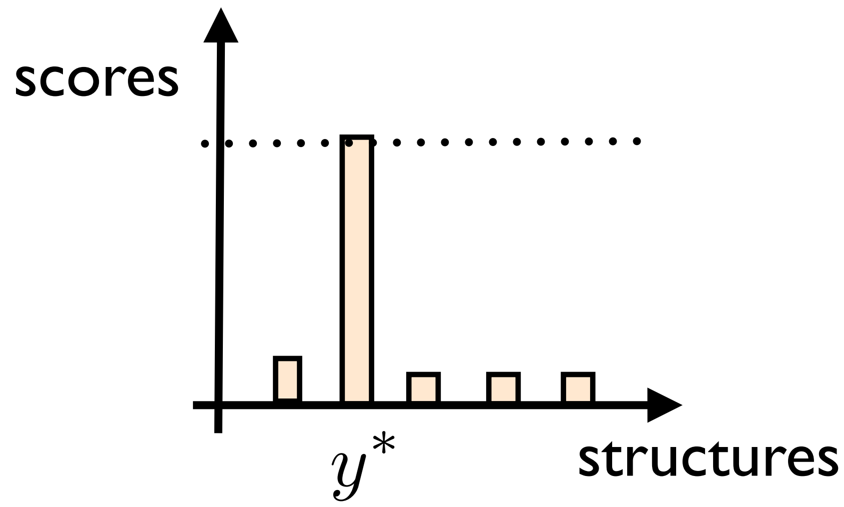


- Randomly perturbing the system reveals its complexity

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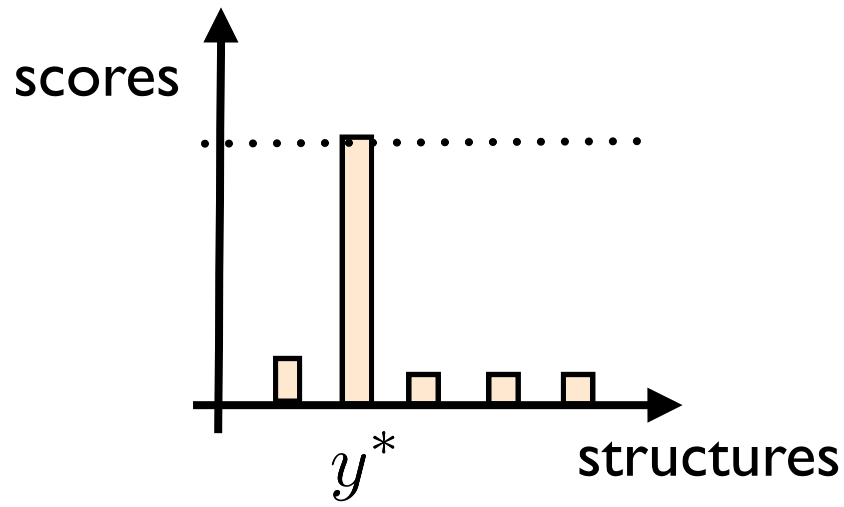


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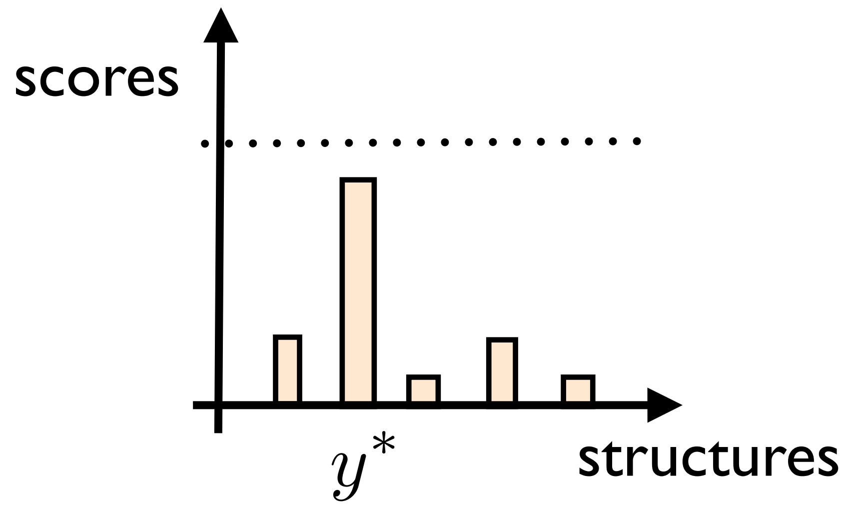


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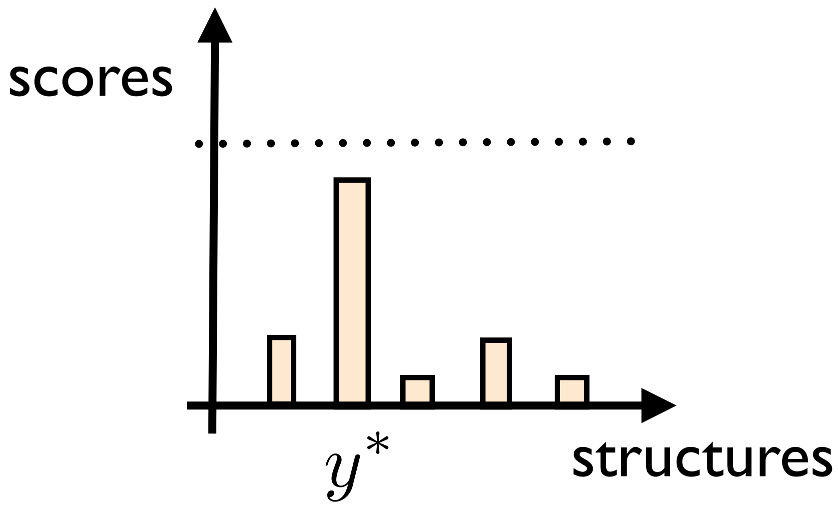


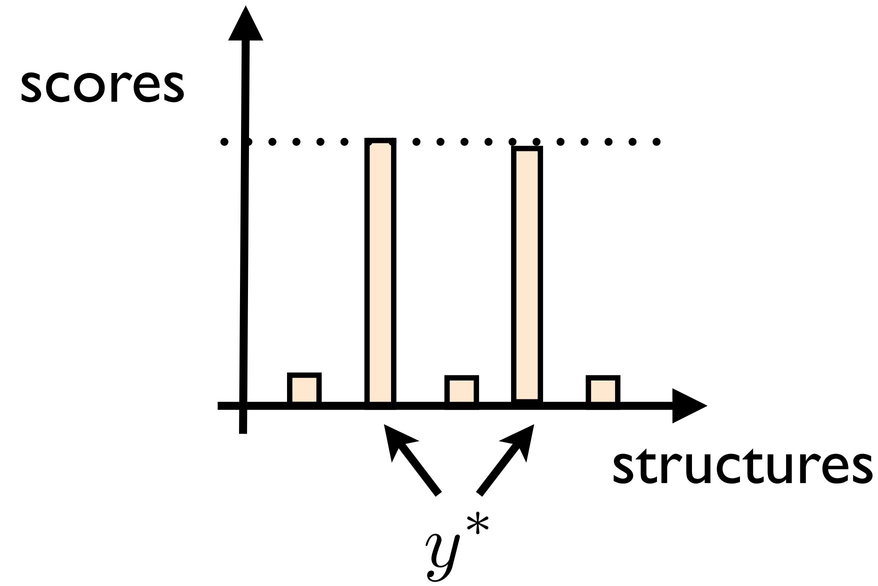


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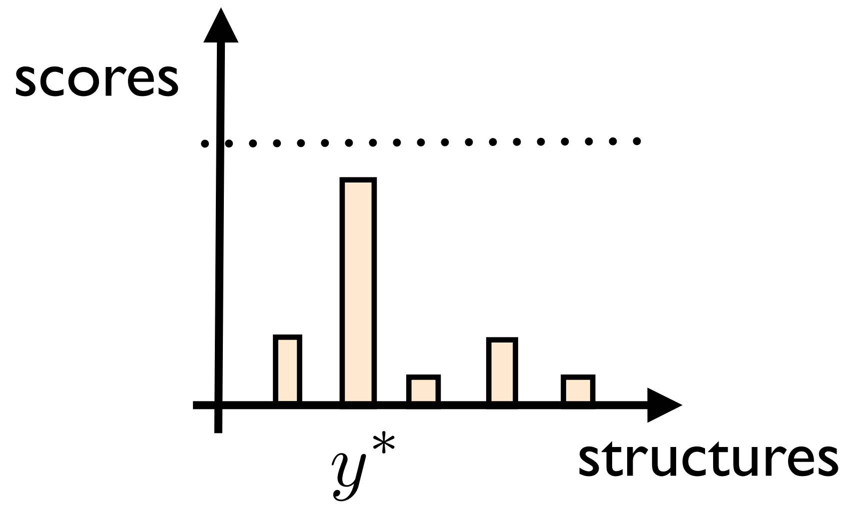


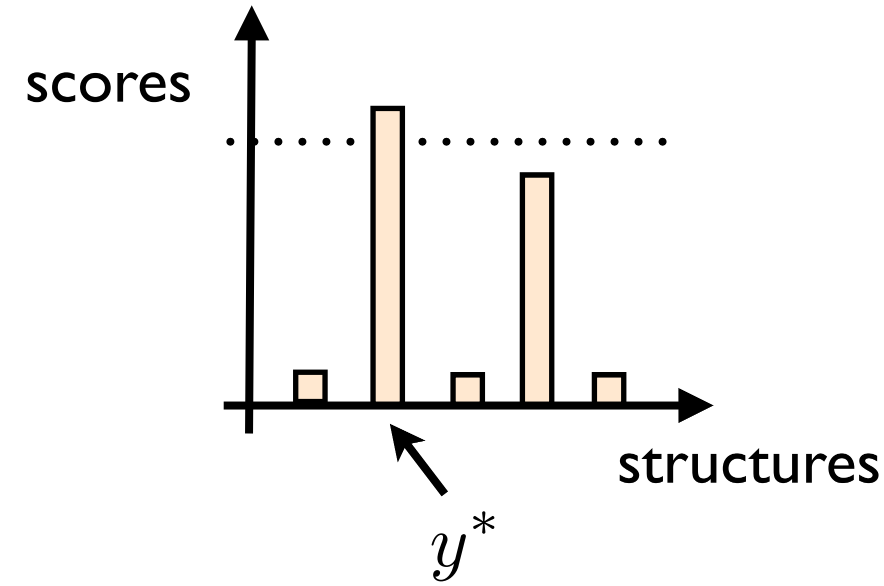


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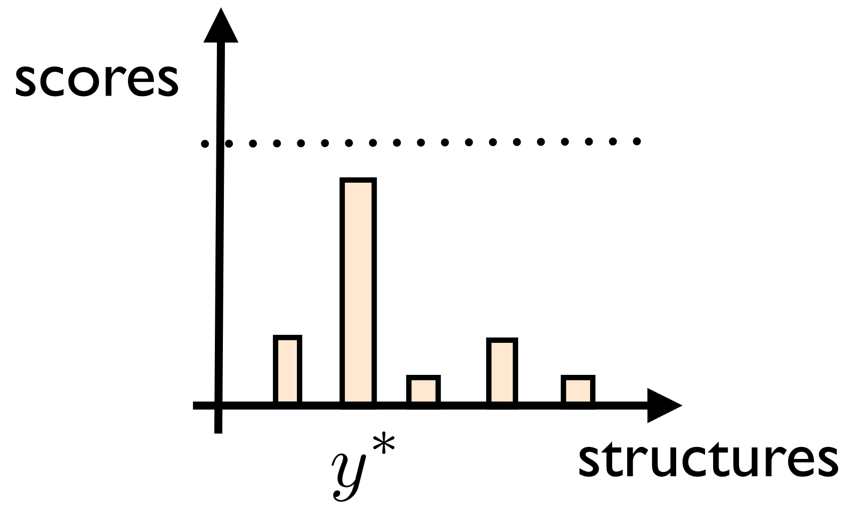


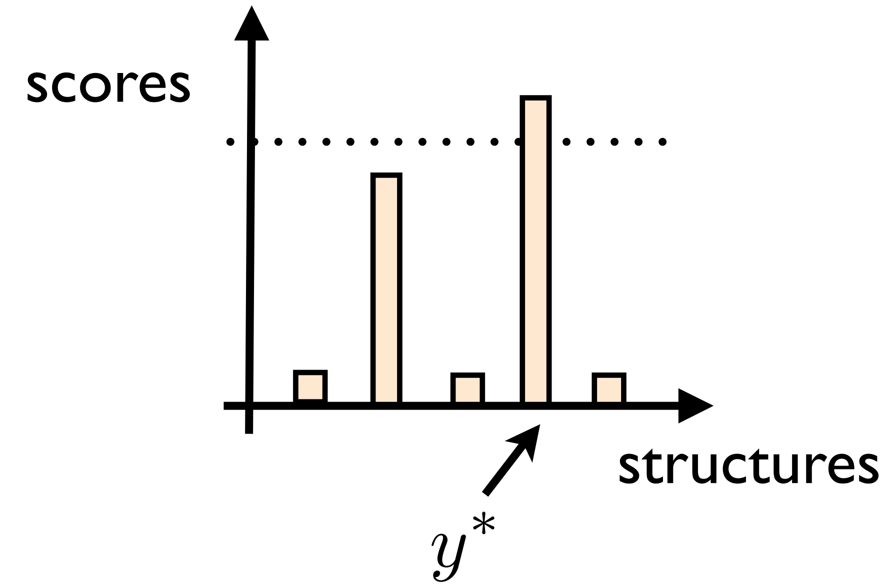


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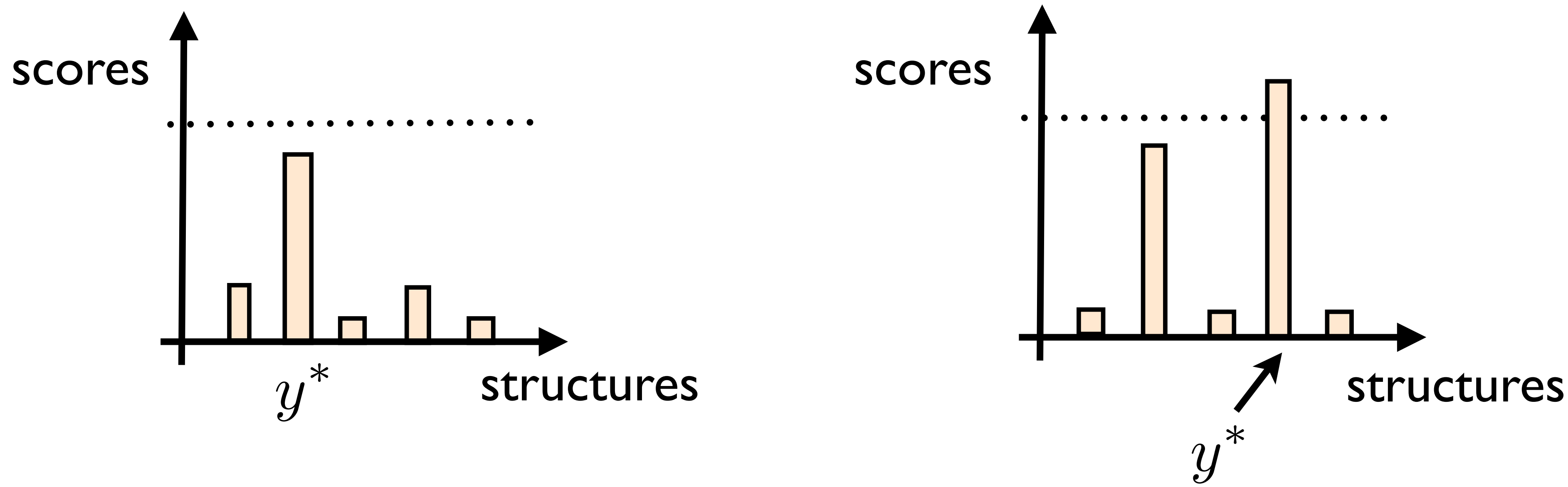


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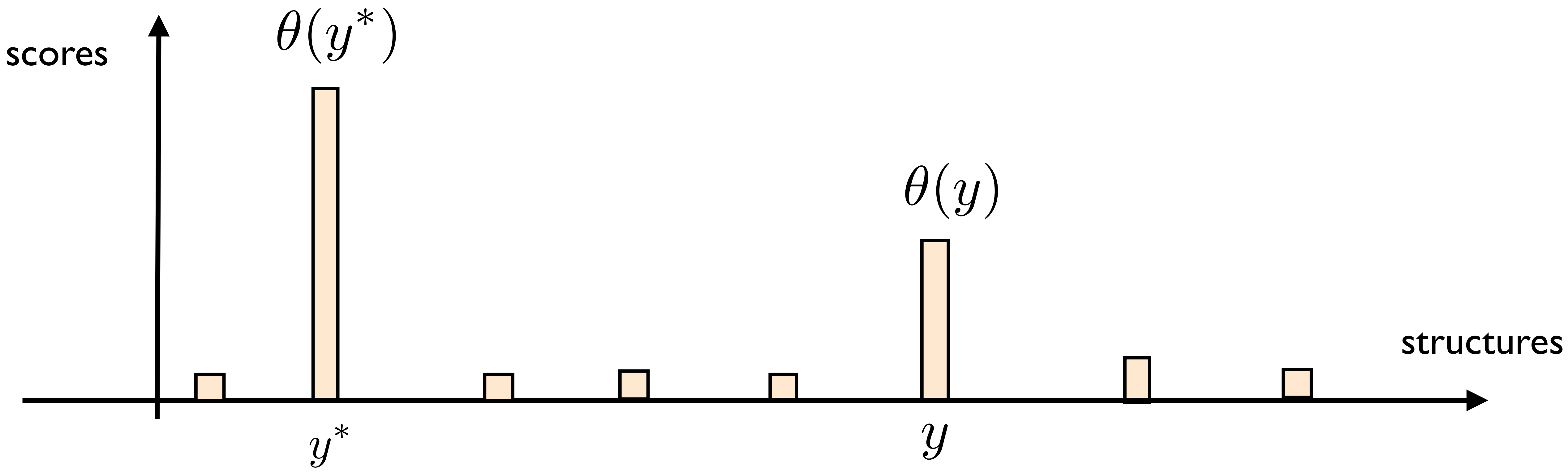
- Related work:
  - McFadden 74 (Discrete choice theory)
  - Talagrand 94 (Canonical processes)

# Random perturbations

- Notation:

scores (potential) $\theta(y)$



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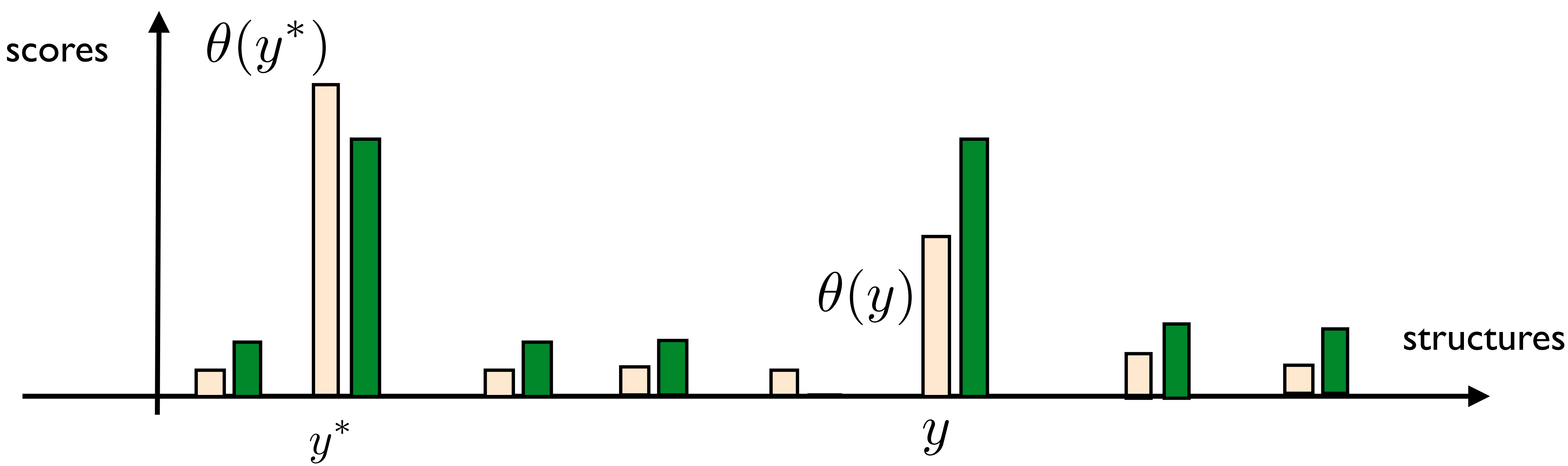

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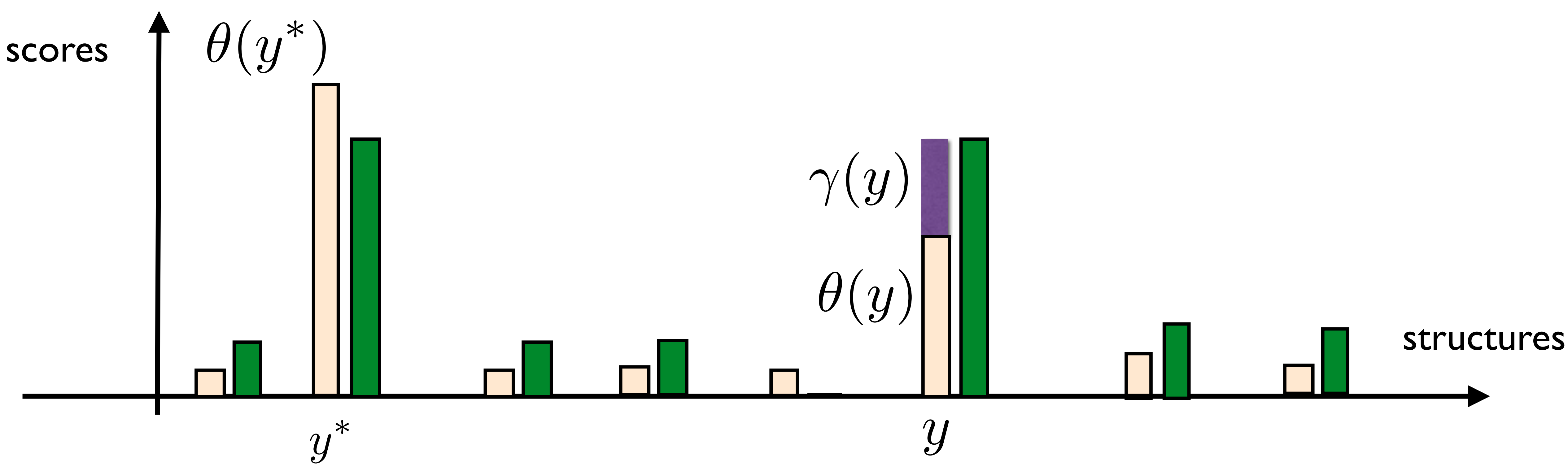

# Random perturbations

- Notation:

| | | |
|---|---|---|
| | scores (potential) | $\theta(y)$ |
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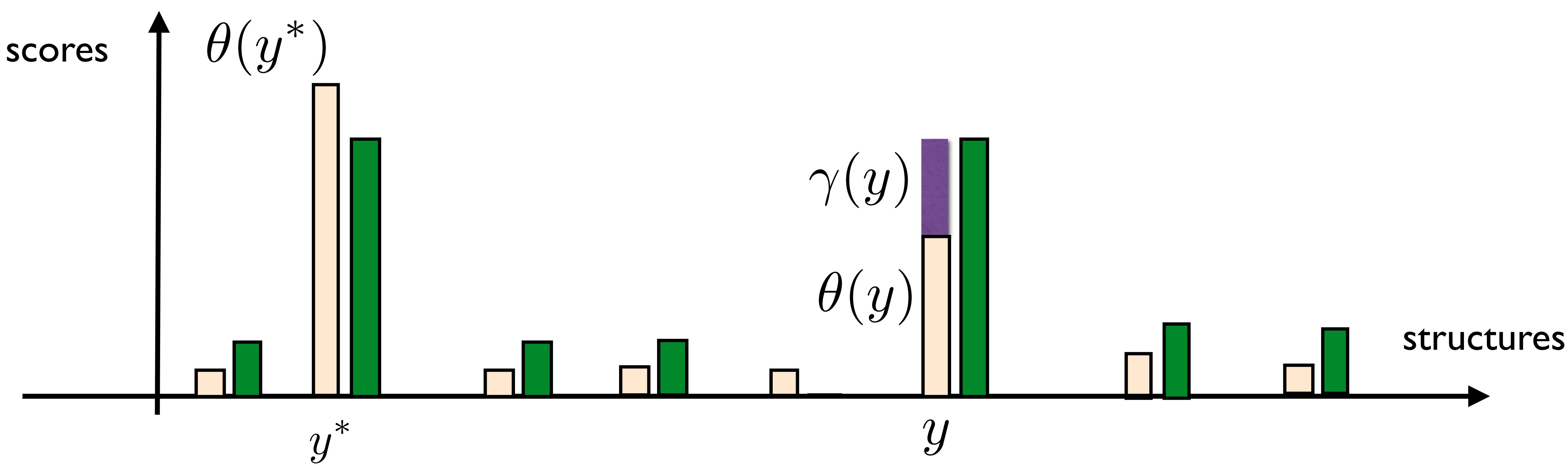
| | perturbations | $\gamma(y)$ |



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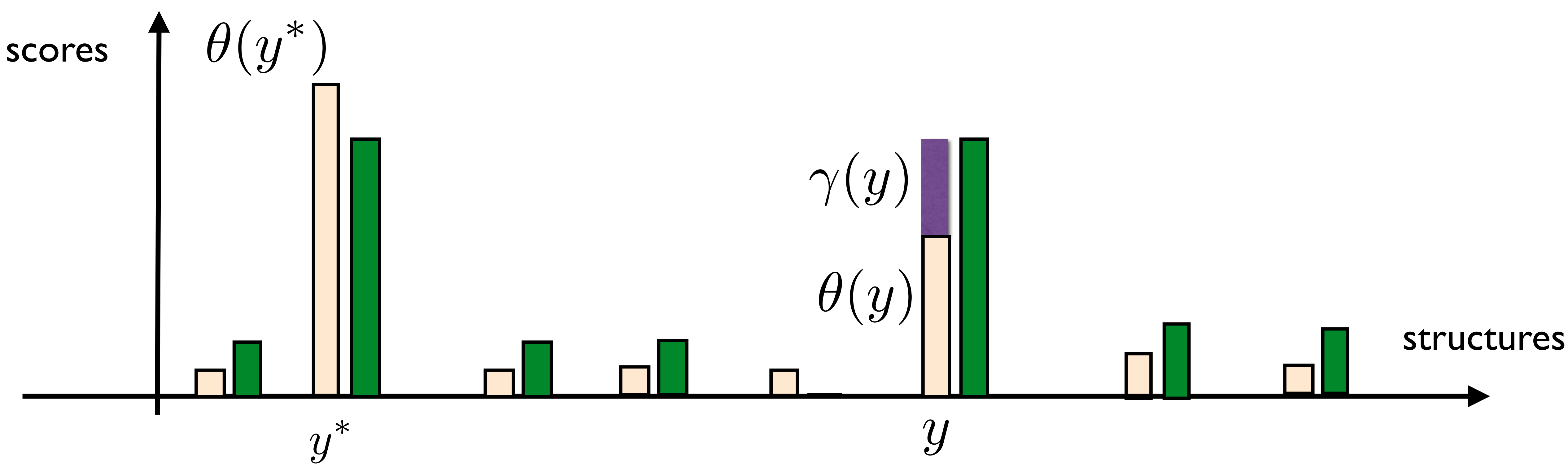
- For every structure y, the perturbation value $\gamma(y)$ is a random variable (y is an index, traditional notation is $\gamma_y$).



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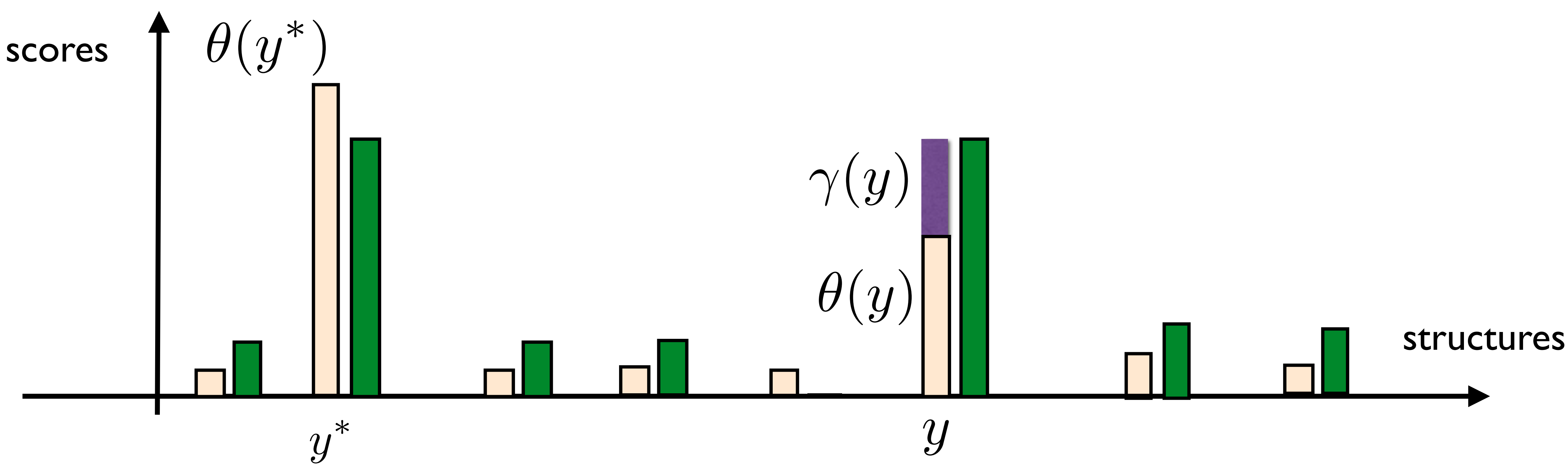
- Perturb-max models: how stable is the maximal structure to random changes in the potential function.



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# Perturb-max models

- **Theorem**

Let $\gamma(y)$ be i.i.d. with Gumbel distribution with zero mean

$$F(t) \stackrel{def}{=} P[\gamma(y) \leq t] = \exp(-\exp(-t))$$

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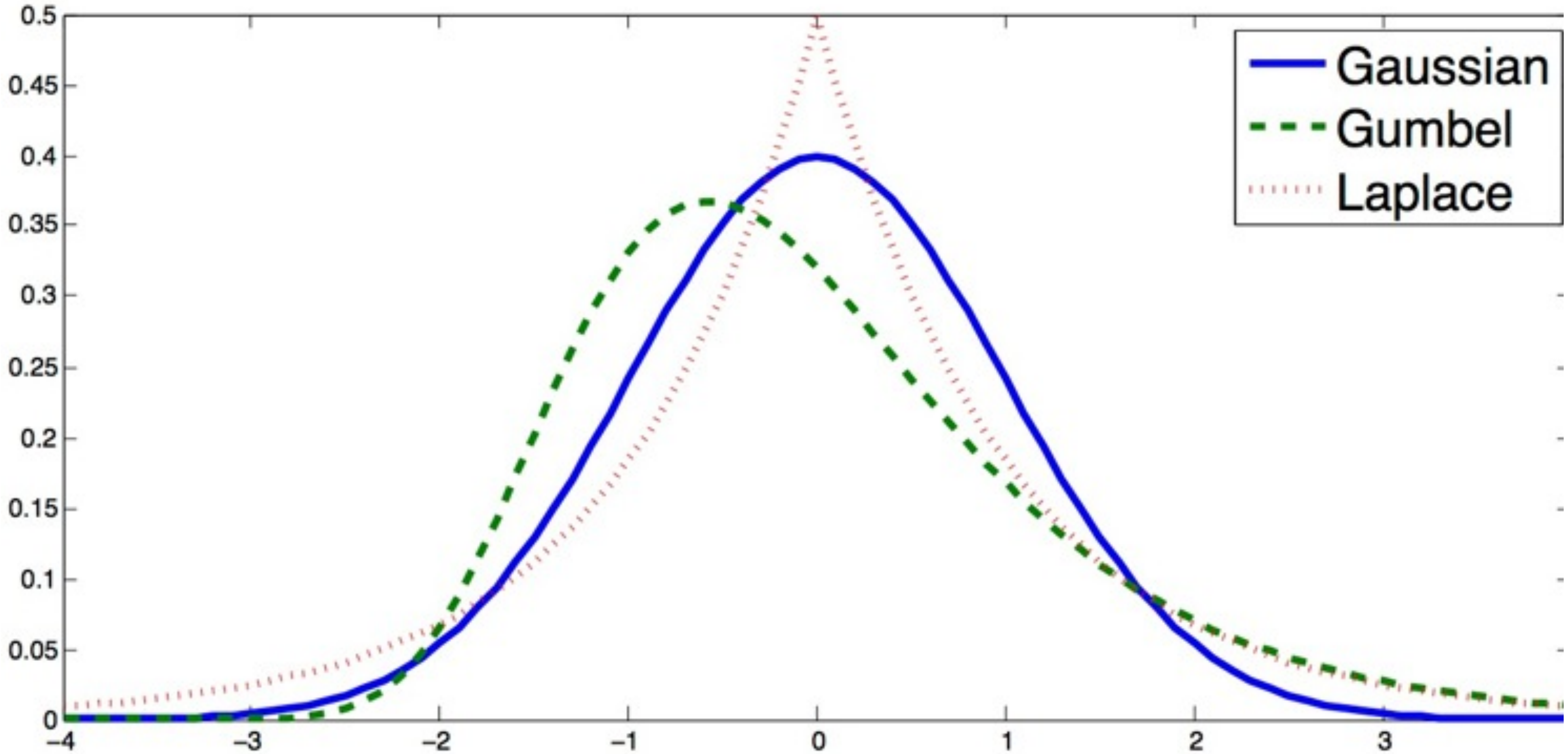
$$f(t) = F'(t) = \exp(-t)F(t)$$



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then the perturb-max model is the Gibbs distribution

$$\frac{1}{Z}\exp(\theta(y)) = P_{\gamma \sim Gumbel}[y = \arg\max_{\hat{y}}\{\theta(\hat{y}) + \gamma(\hat{y})\}]$$

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$$= \exp(-\sum_y \exp(-(t - \theta(y)))) = F(t - \log Z)$$

# Perturb-max models

- Max stability:

$$\log\Big(\sum_y \exp(\theta(y))\Big) = E_{\gamma \sim Gumbel}\Big[\max_y\{\theta(y) + \gamma(y)\}\Big]$$

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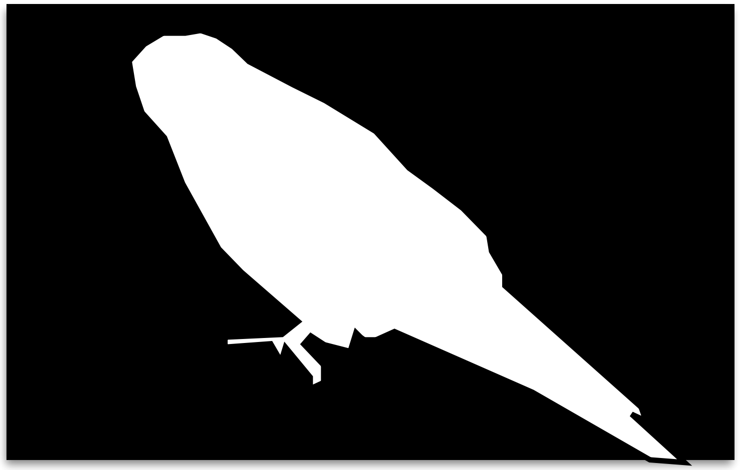

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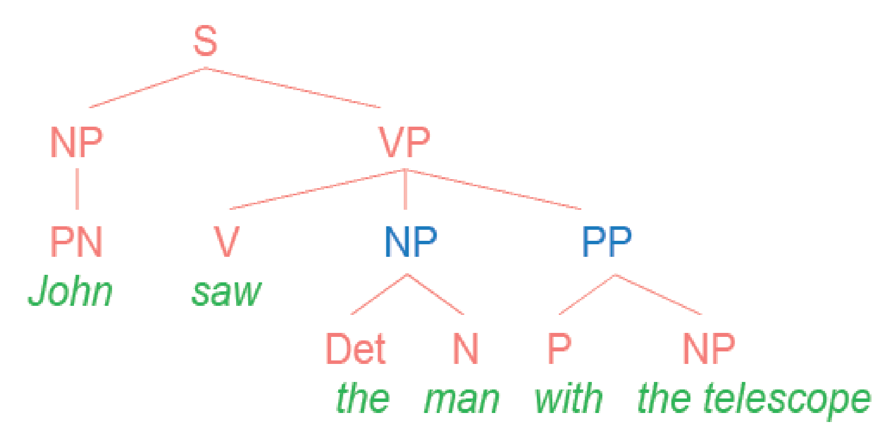

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- Use low dimension perturbations (Papandreou & Yuille 11, Tarlow et al. 12, Hazan & Jaakkola 12)

$$P_\gamma[y = \arg\max_{\hat{y}}\{\theta(\hat{y}) + \sum_{i=1}^{n} \gamma_i(\hat{y}_i)\}]$$

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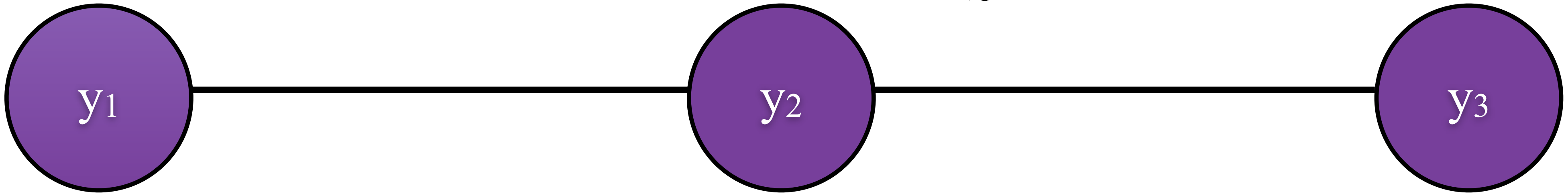

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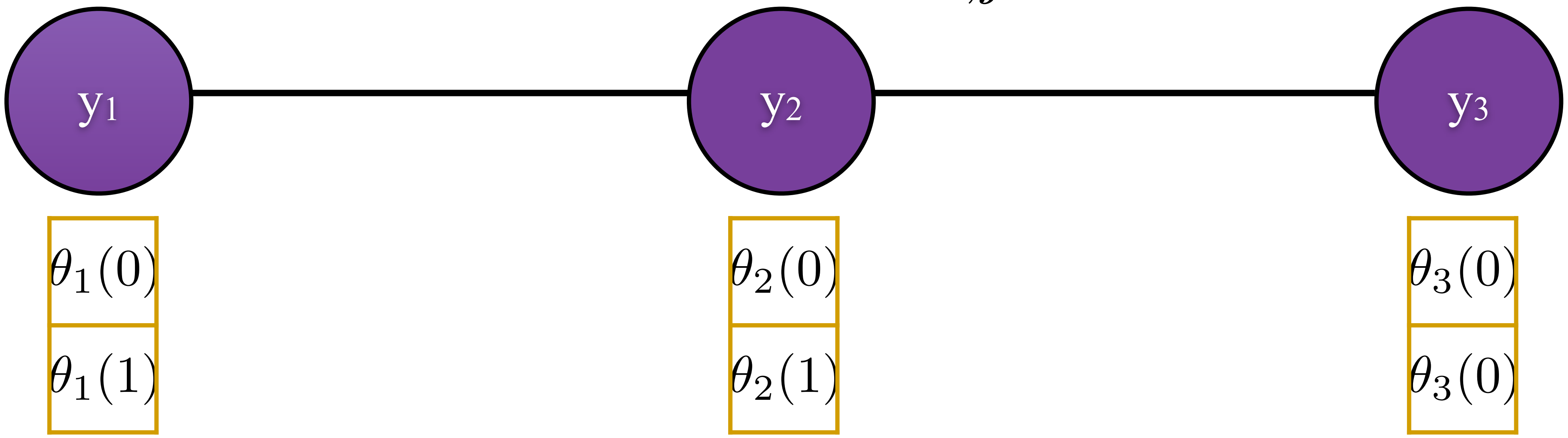

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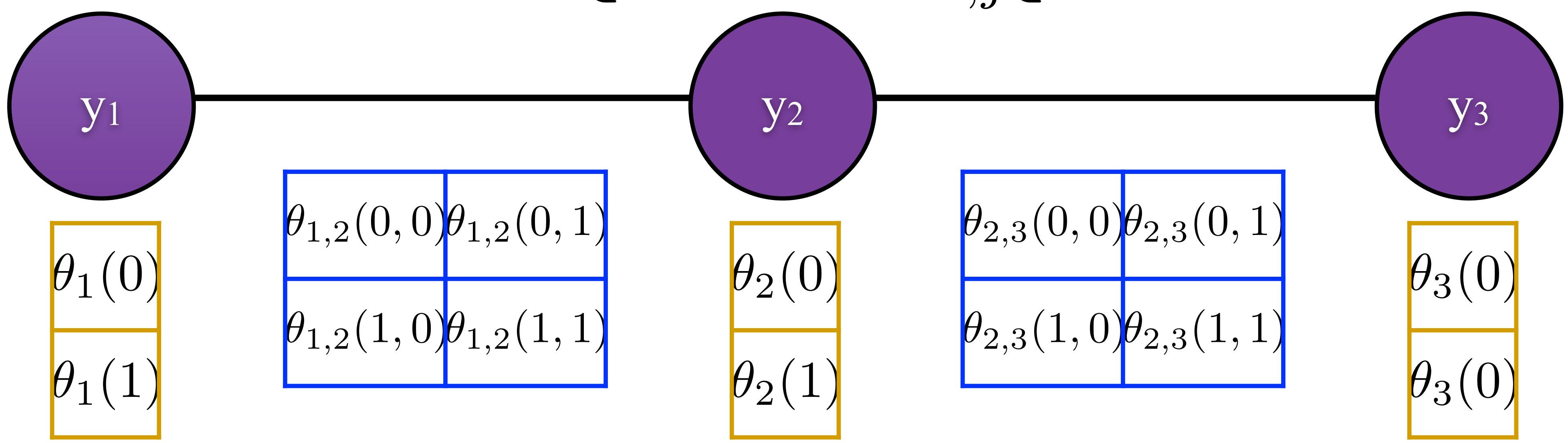

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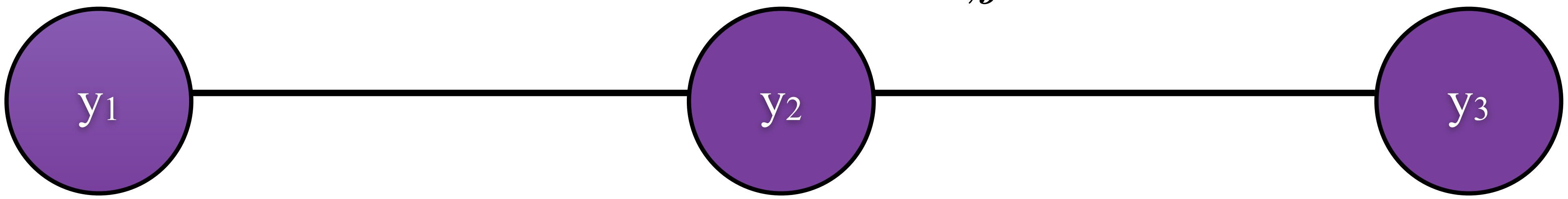


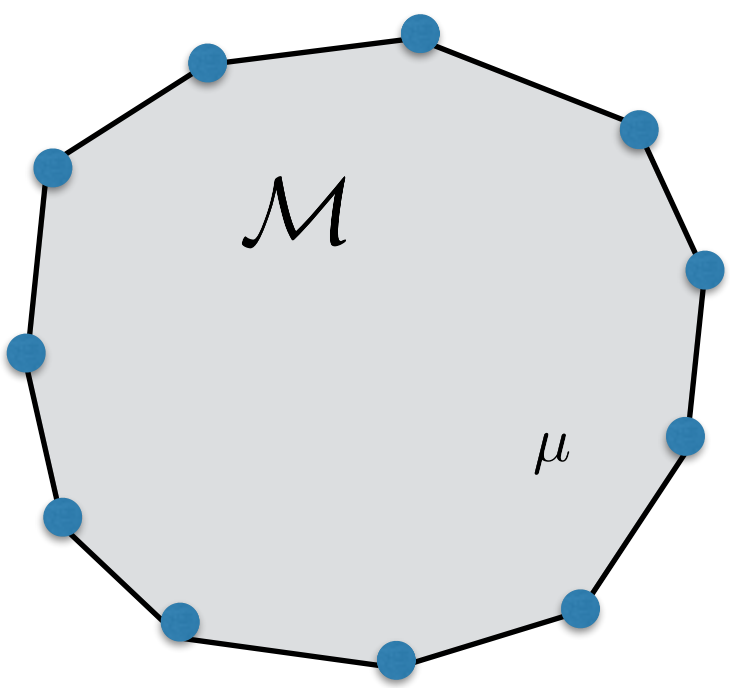

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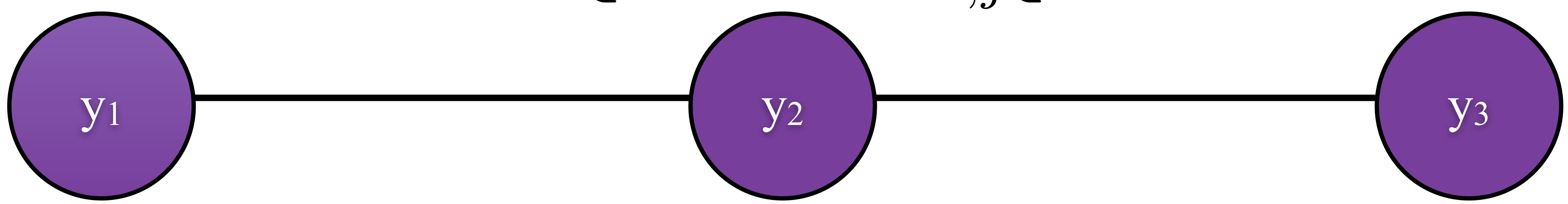


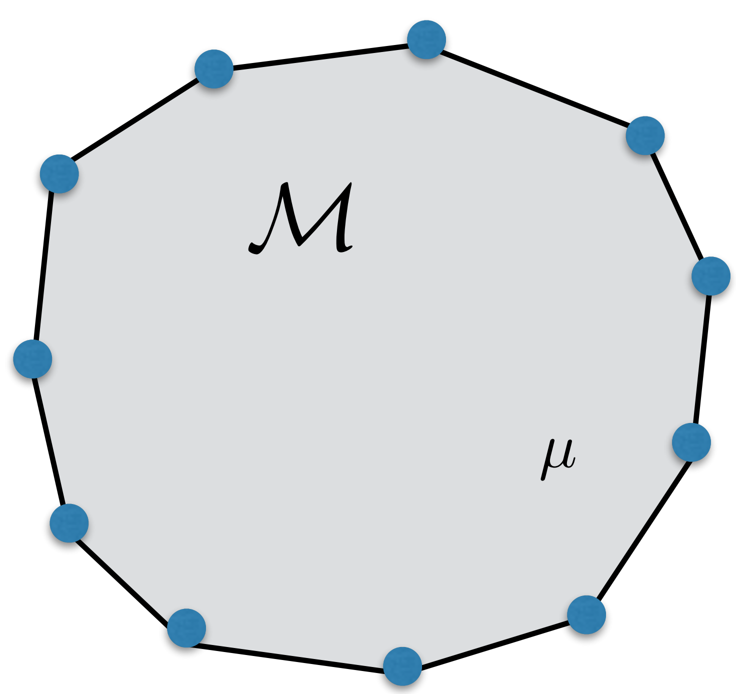


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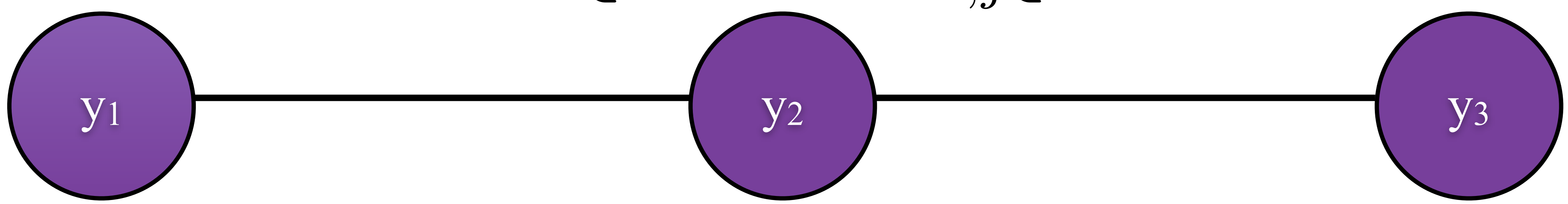


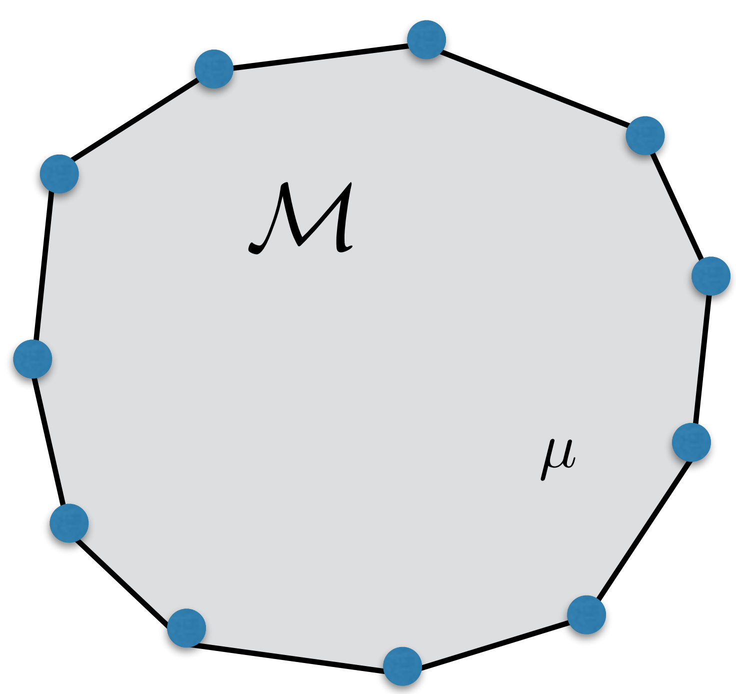


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$$\exists p(y_1, y_2, y_3) \text{ s.t. } \quad \mu_1(y_1) = \sum_{y_2, y_3} p(y_1, y_2, y_3), \ldots$$

$$\mu_{1,2}(y_1, y_2) = \sum_{y_3} p(y_1, y_2, y_3), \ldots$$

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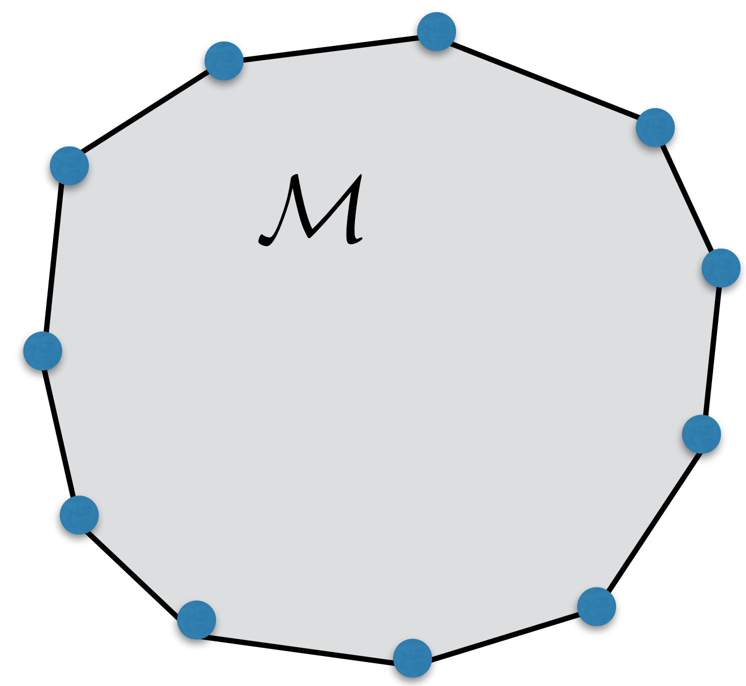

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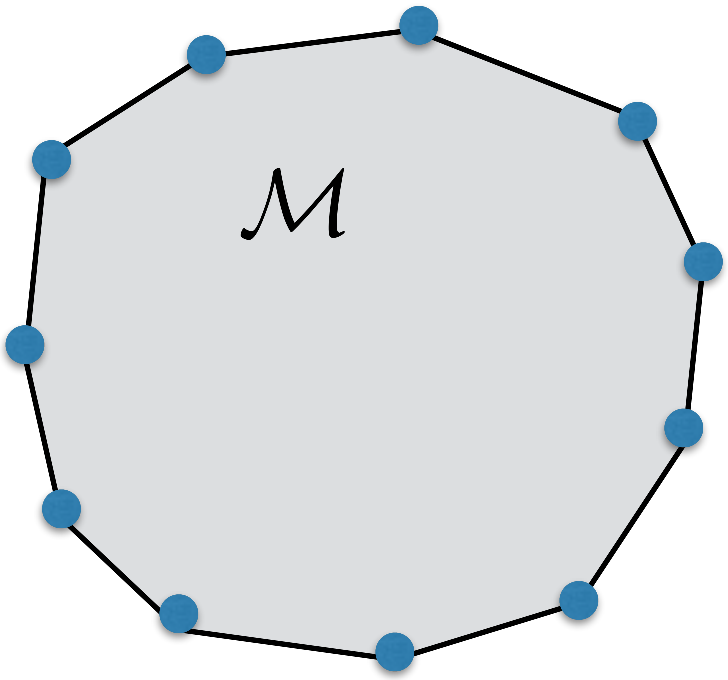

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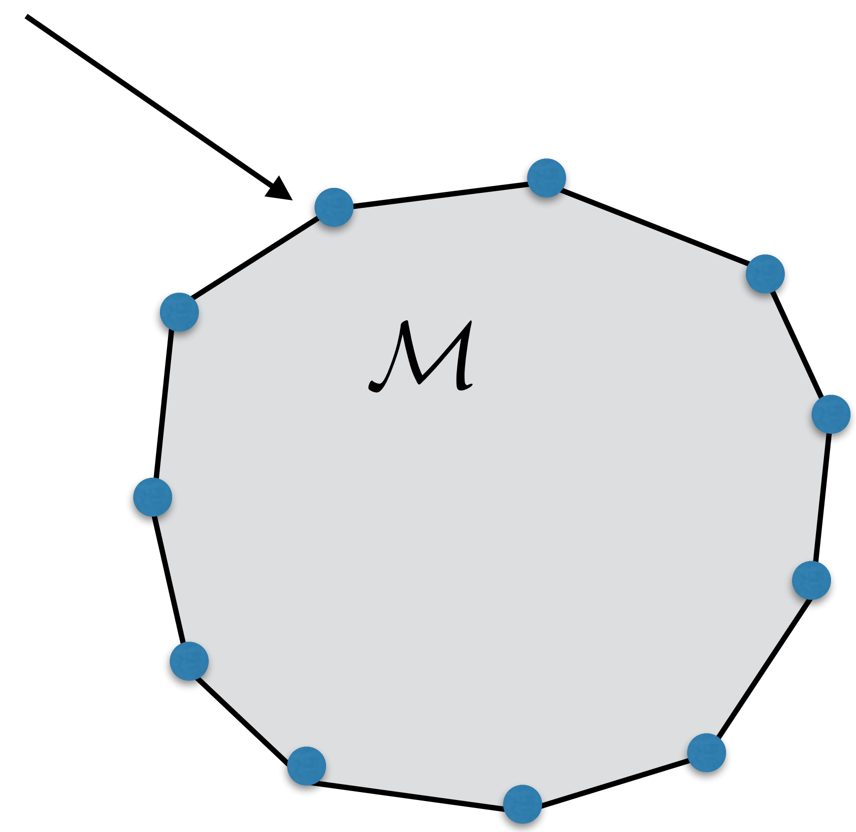

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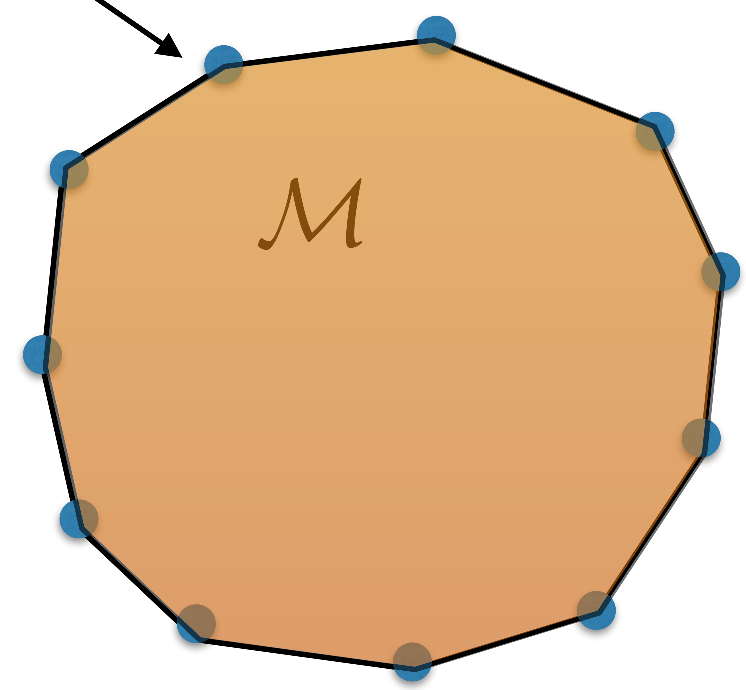

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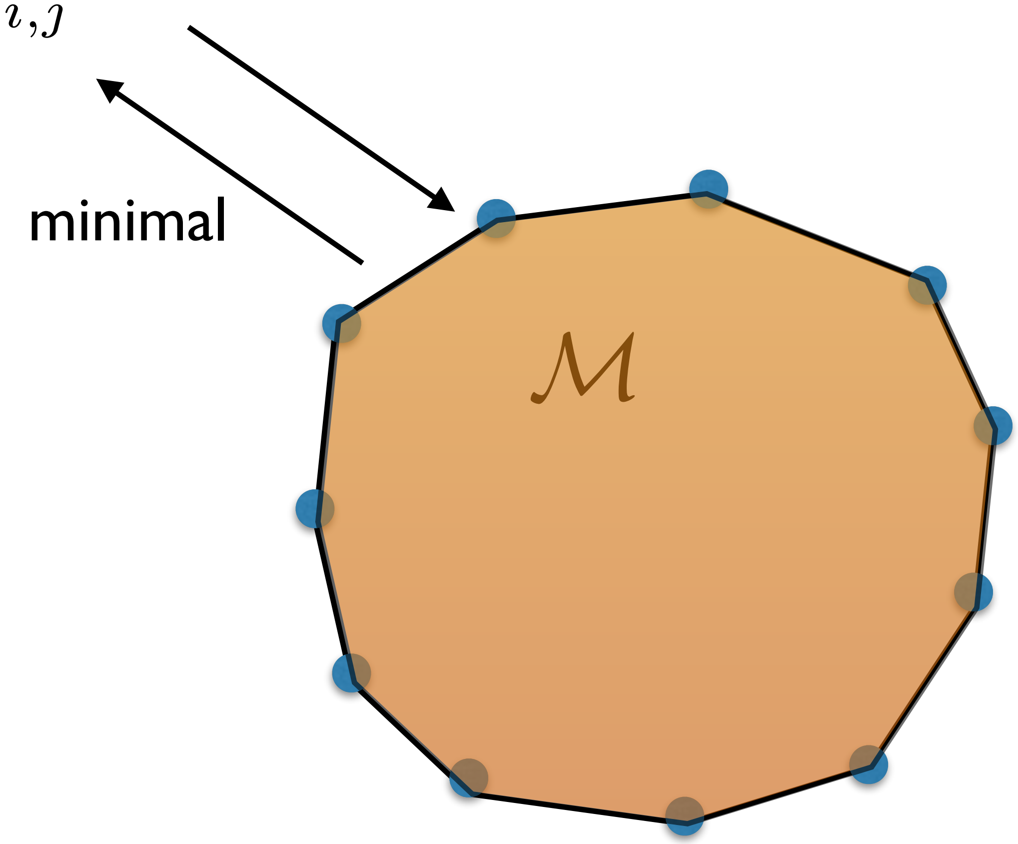

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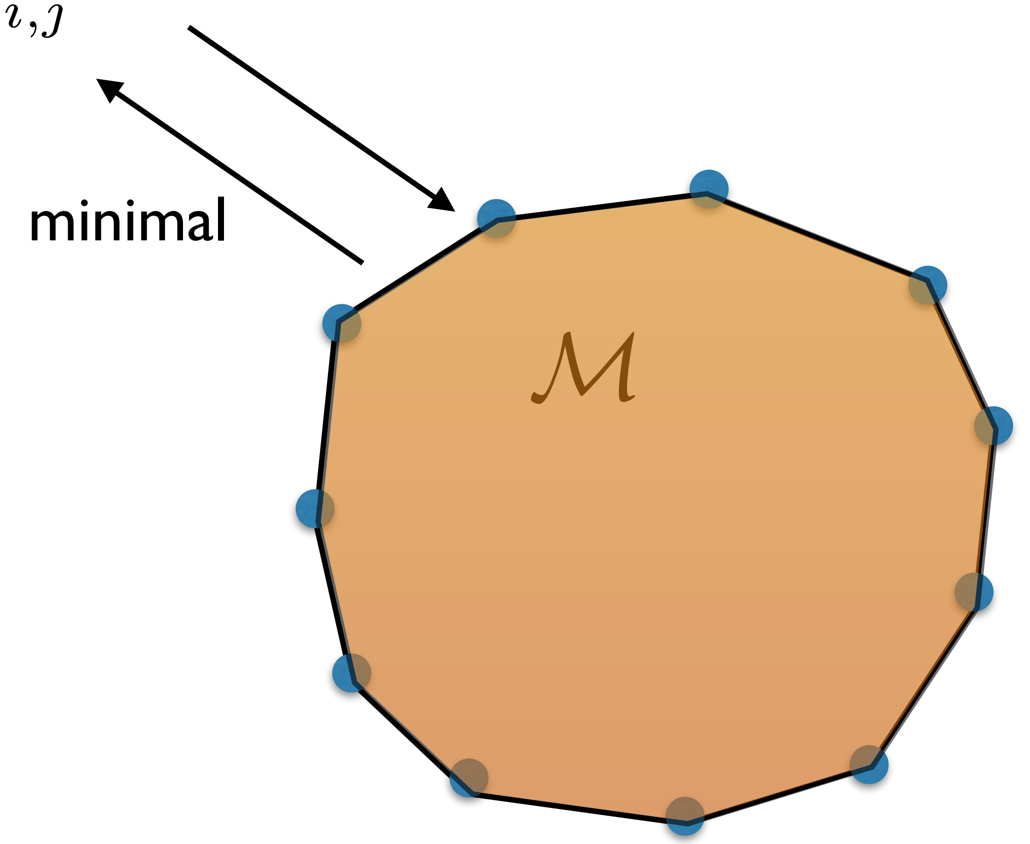


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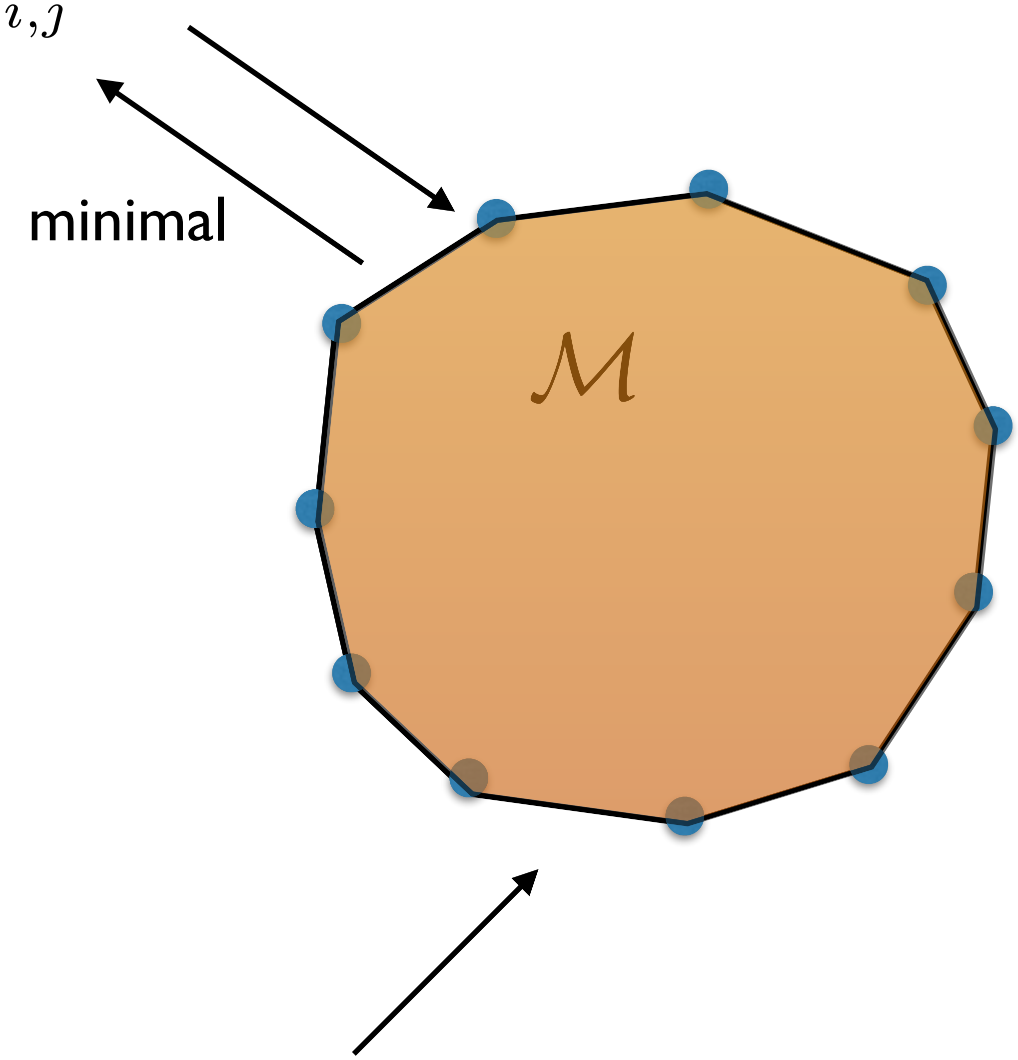


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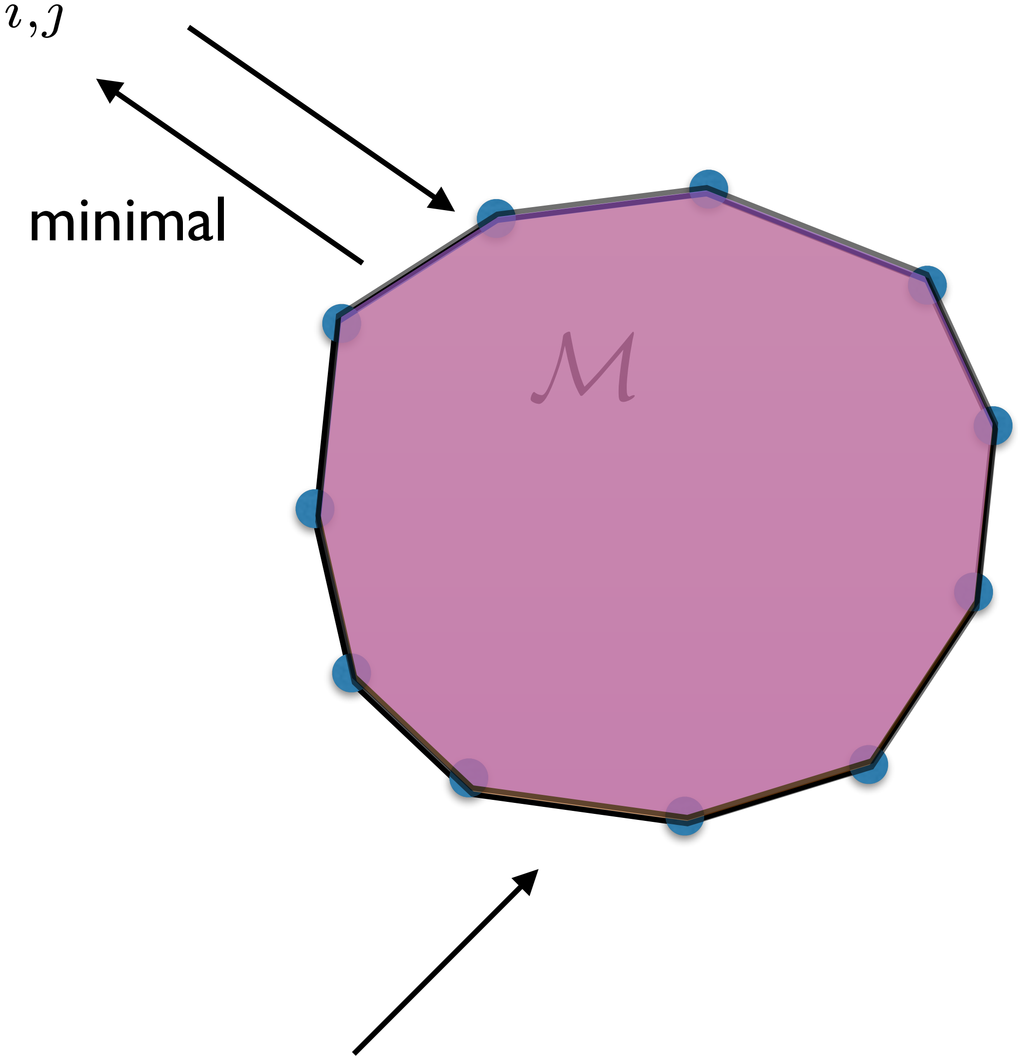


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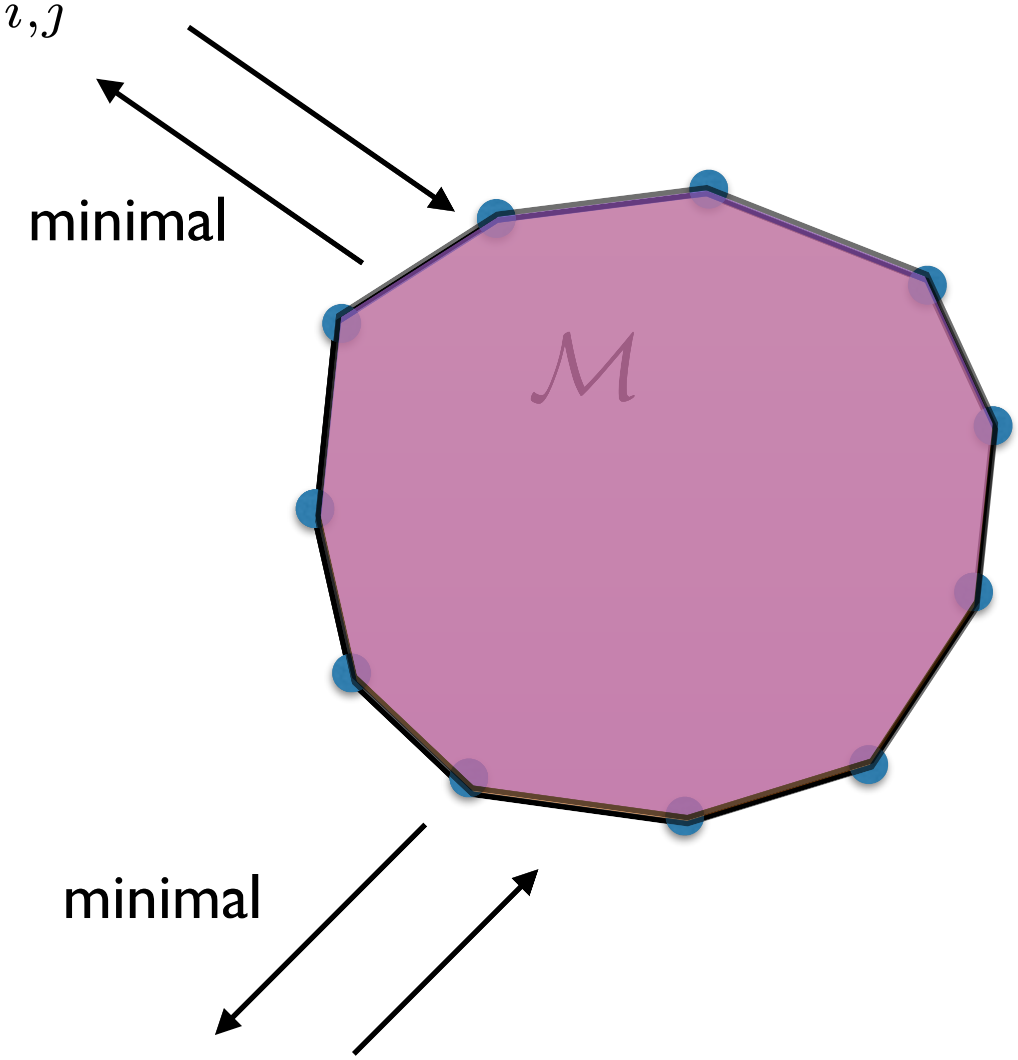


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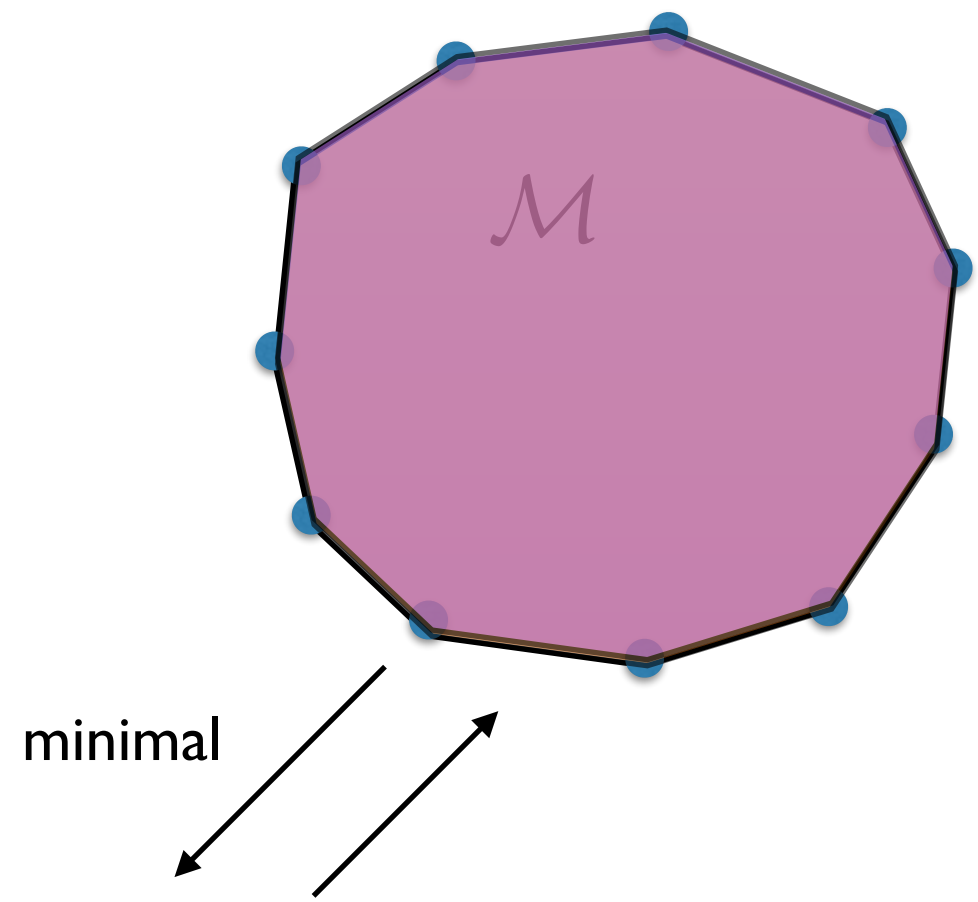


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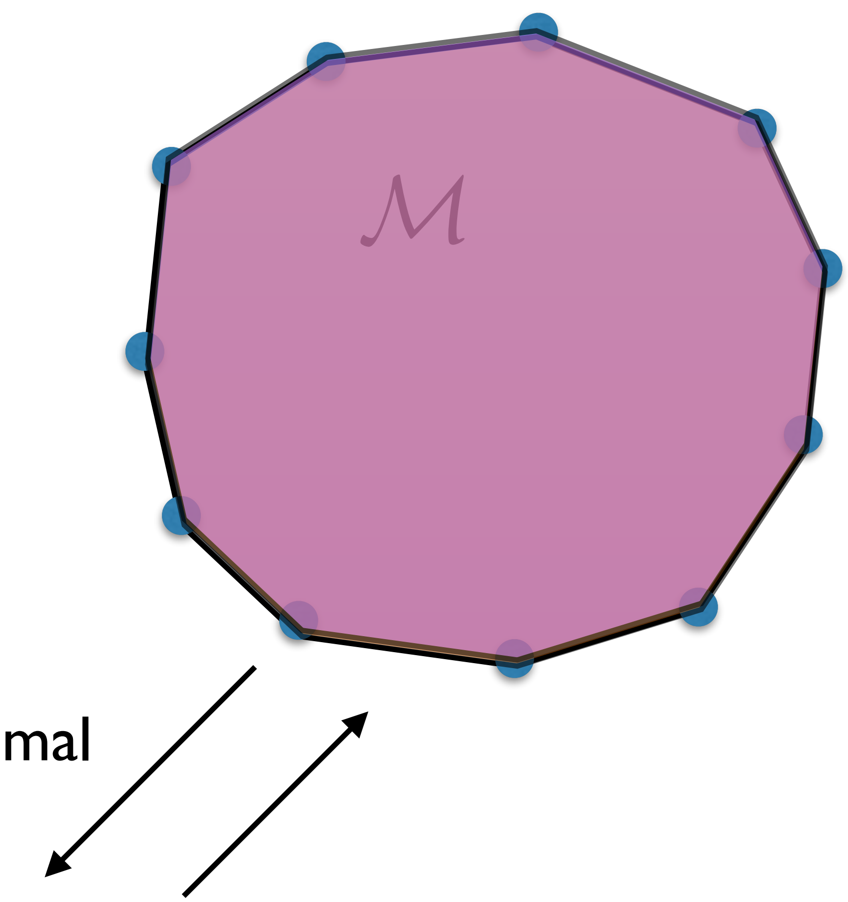


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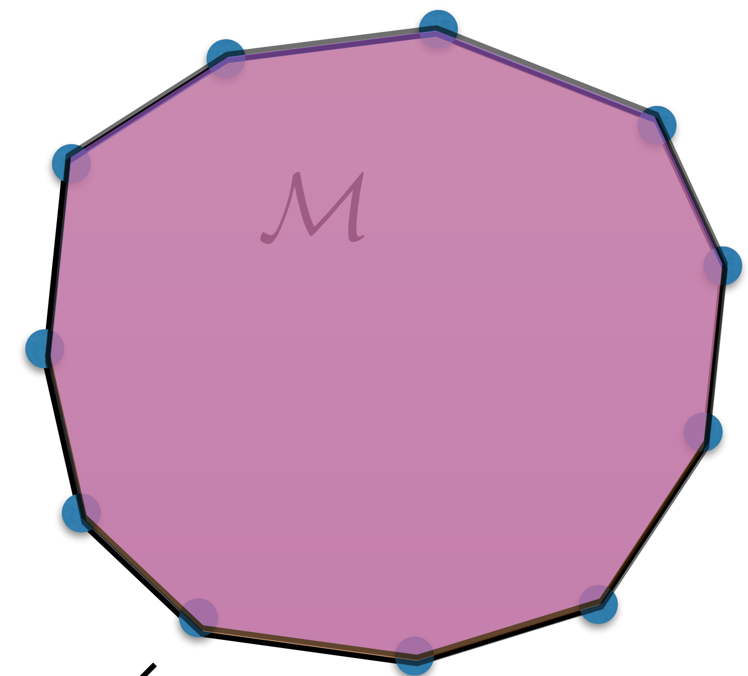


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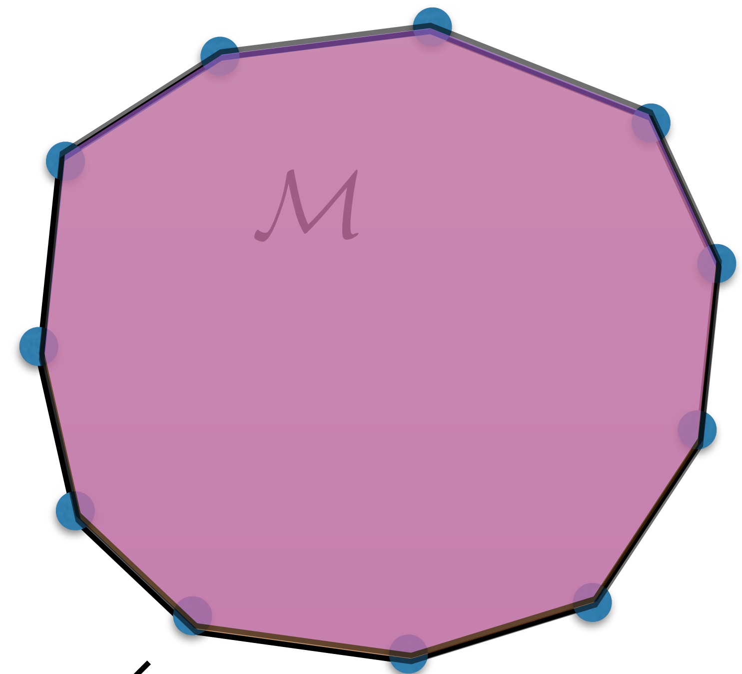


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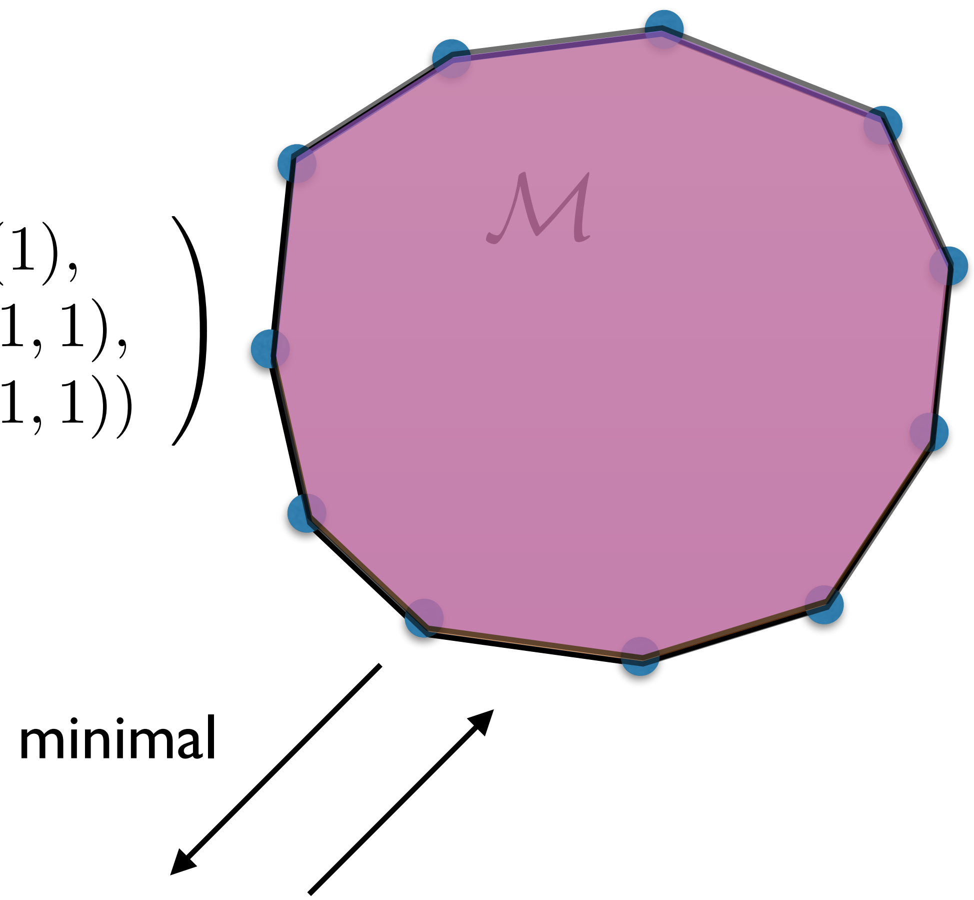


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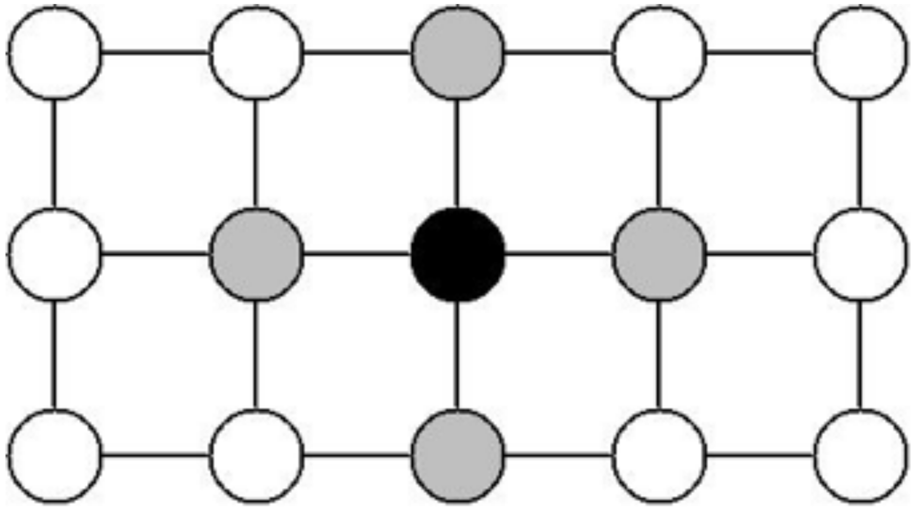

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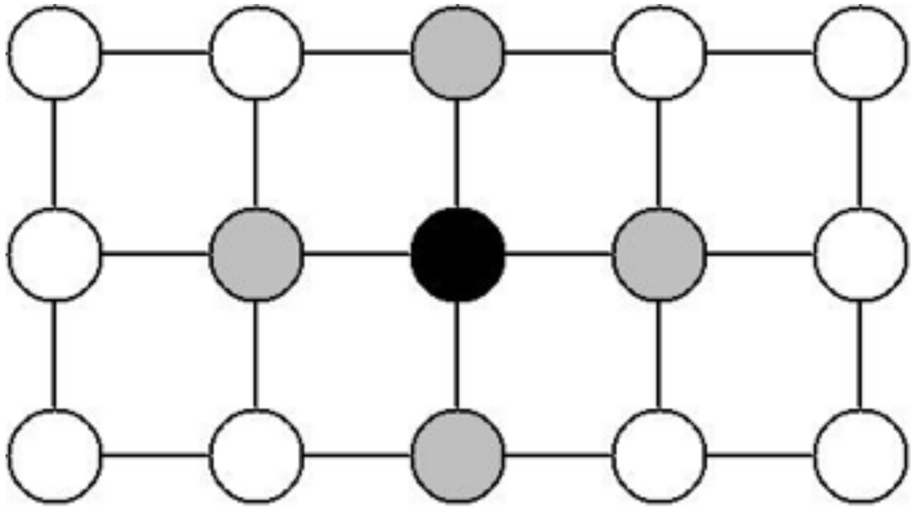

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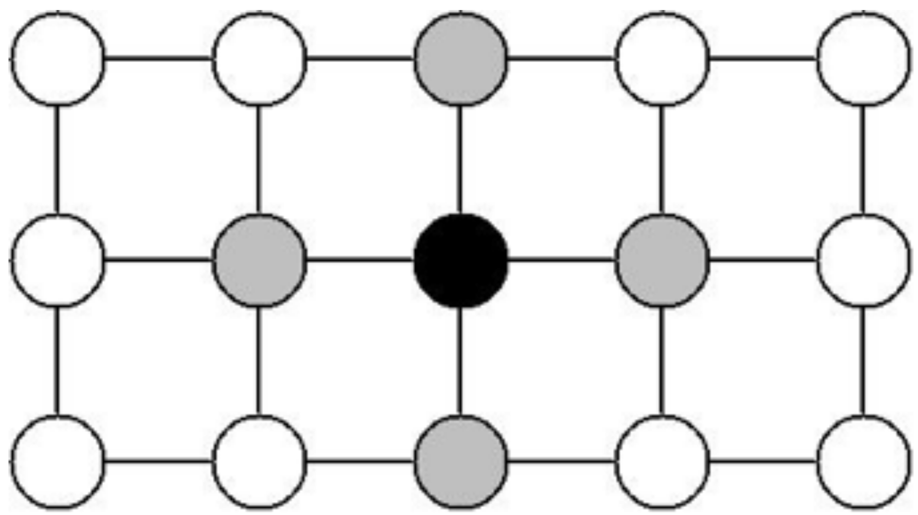

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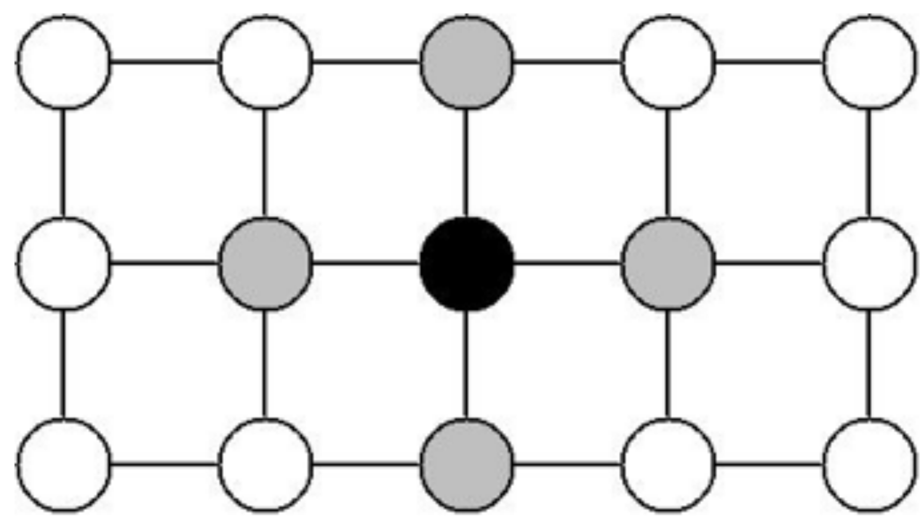

- What is the modeling power of perturb-max models ?
- Can they model dependencies beyond graph neighborhoods?

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- Perturb-max can model these long range dependencies (e.g., legs / arms dependencies).

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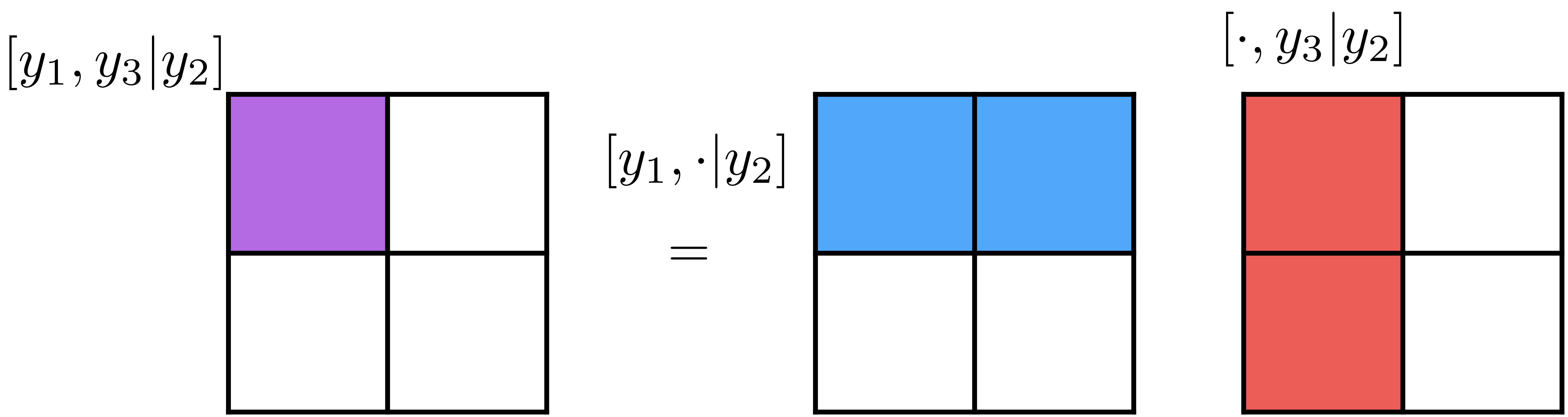

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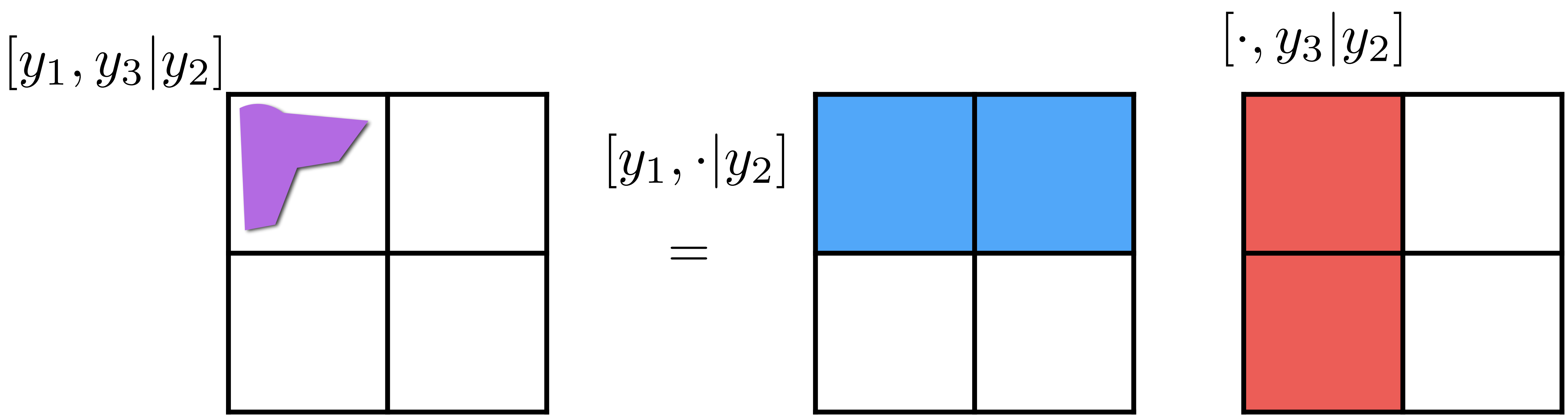

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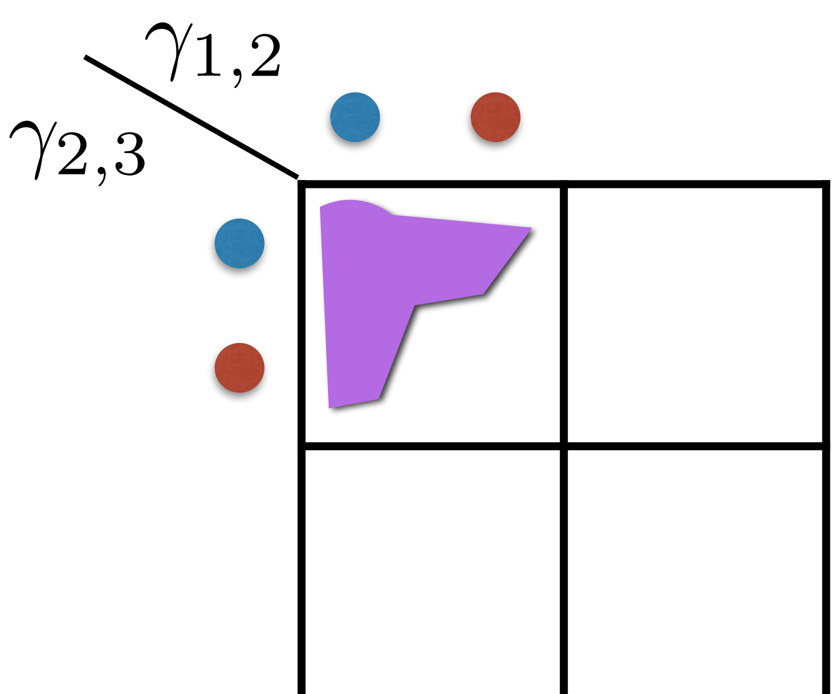


high

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$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} \quad \theta_{i,j} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

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- General notation:

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smooth (vanishing gradient ⇒ optimum)

- Key fact: the perturb-max models are convolutions

$$p(y; w) = P_{\gamma \sim q_w}\left[ y = \arg\max_{\hat{y}} \left\{ \sum_\alpha \theta_\alpha(\hat{y}_\alpha; \gamma_\alpha) \right\} \right]$$

$$= \int \underbrace{q_w(\gamma)}_{\text{smooth}} \underbrace{1\left[y = \arg\max_{\hat{y}} \{ \sum_\alpha \theta_\alpha(\hat{y}_\alpha; \gamma_\alpha) \}\right]}_{\text{non-smooth}} d\gamma$$

# Log-likelihood learning

- Learn the parameters $w$ that maximize the perturb-max likelihood

$$\max_w \sum_{y \in S} \log p(y; w)$$

- Gradients match between prior and posterior predictions

$$\frac{\partial \sum_{y \in S} \log p(y; w)}{\partial w} = \sum_{y \in S} \left( E\left[ \gamma \;\middle|\; y = \arg\max_{\hat{y}} \sum_{\alpha} \theta_\alpha(\hat{y}_\alpha; \gamma_\alpha) \right] - w \right)$$

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- When using log-concave perturbations the deviation of sampled average from the expectation decays exponentially (Orabona, Hazan, Sarwate, Jaakkola 14)

# Outline

- Random perturbation - why and how?
  - Sampling likely structures as fast as finding the most likely one.
- Connections and Alternatives to Gibbs distribution:
  - the marginal polytope
  - the modeling power of perturb-max models
- Learning with perturb-max models
  - log-likelihood learning
  - interactive learning using new entropy bounds
  - online learning
  - loss minimization and PAC-Bayesian bounds

# Interactive learning

- Use perturb-max models to sample possible annotations and estimate user clicks [Maji, Hazan, Jaakkola 14]

# Interactive learning

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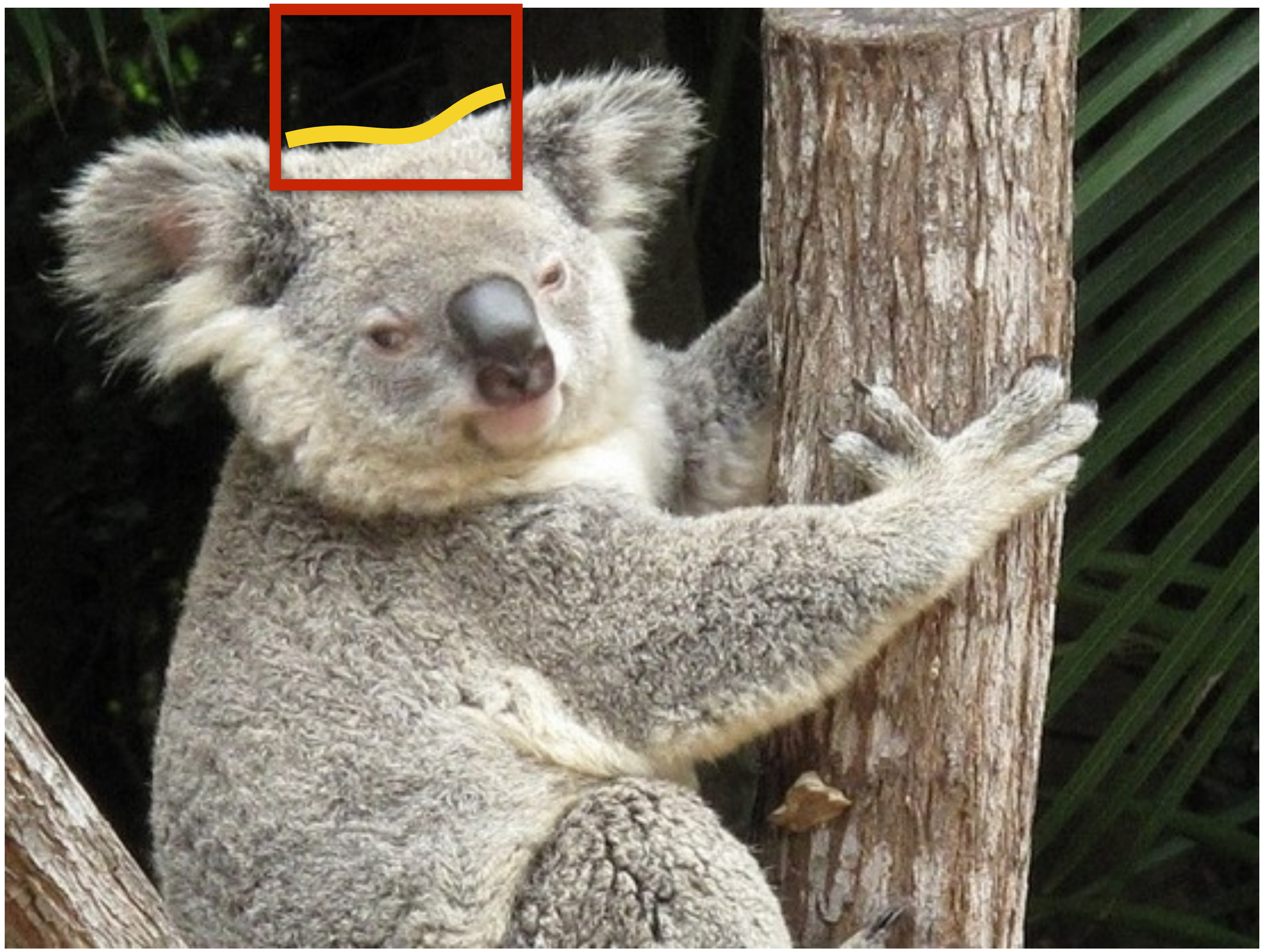

# Interactive learning

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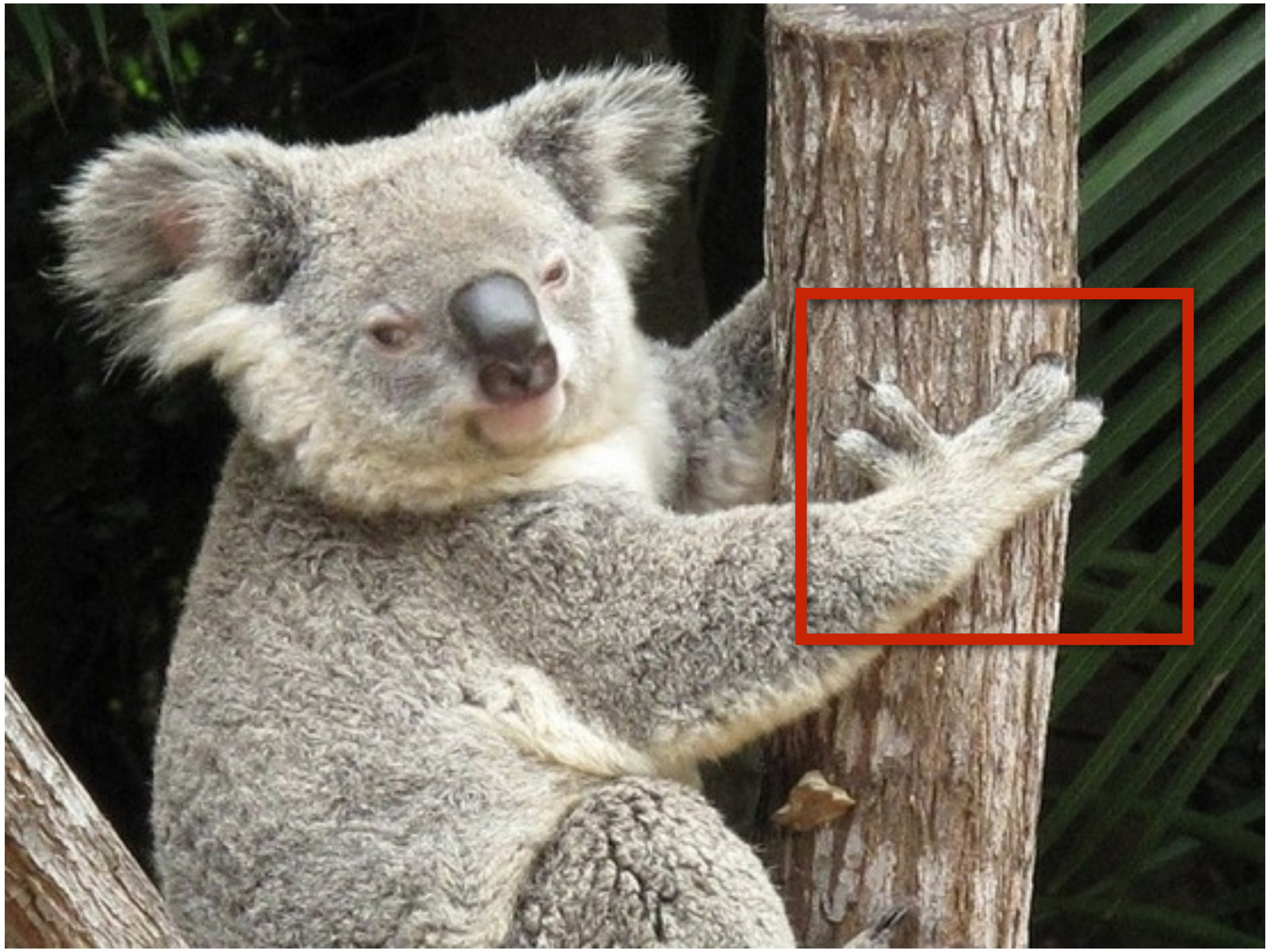
# Interactive learning

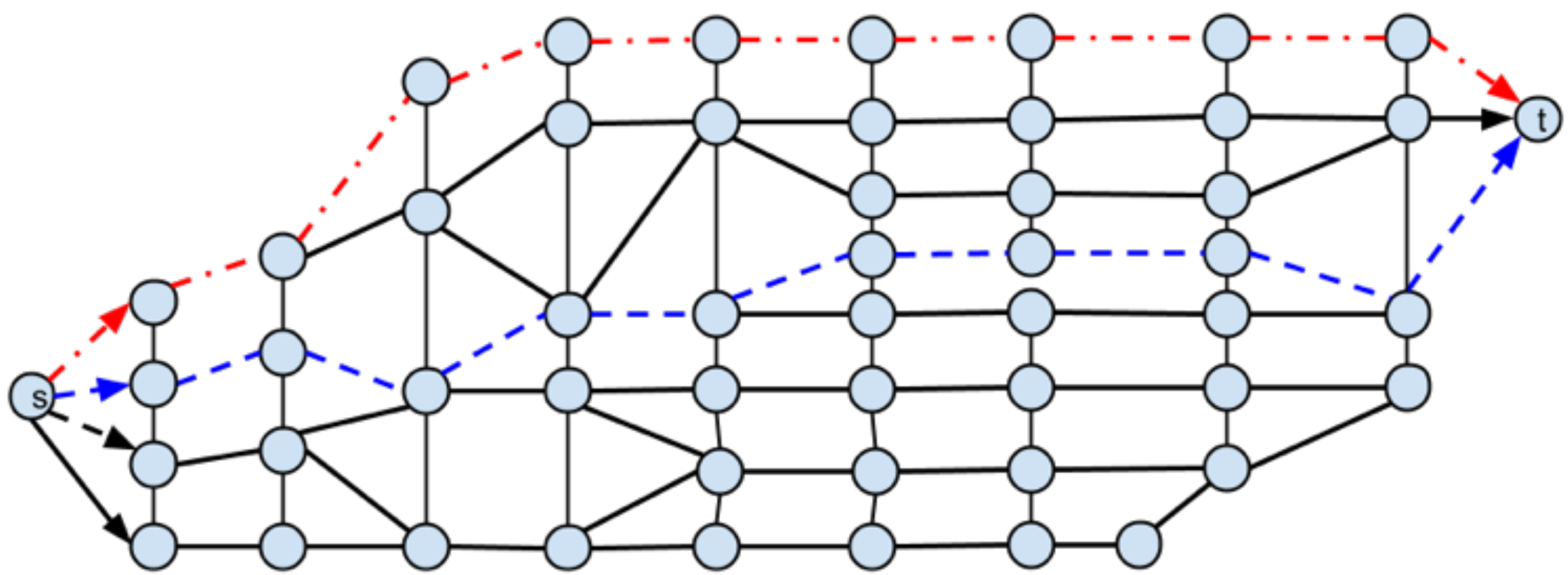
- Choose an area with the largest uncertainty reduction

$$U(p_\theta) = E_\gamma\Big[\sum_{i=1}^{n} \gamma_i(y_i^*)\Big] \qquad y^* = \arg\max_{\hat{y}}\{\theta(\hat{y}) + \sum_{i=1}^{n} \gamma_i(\hat{y}_i)\}$$



# Online Learning

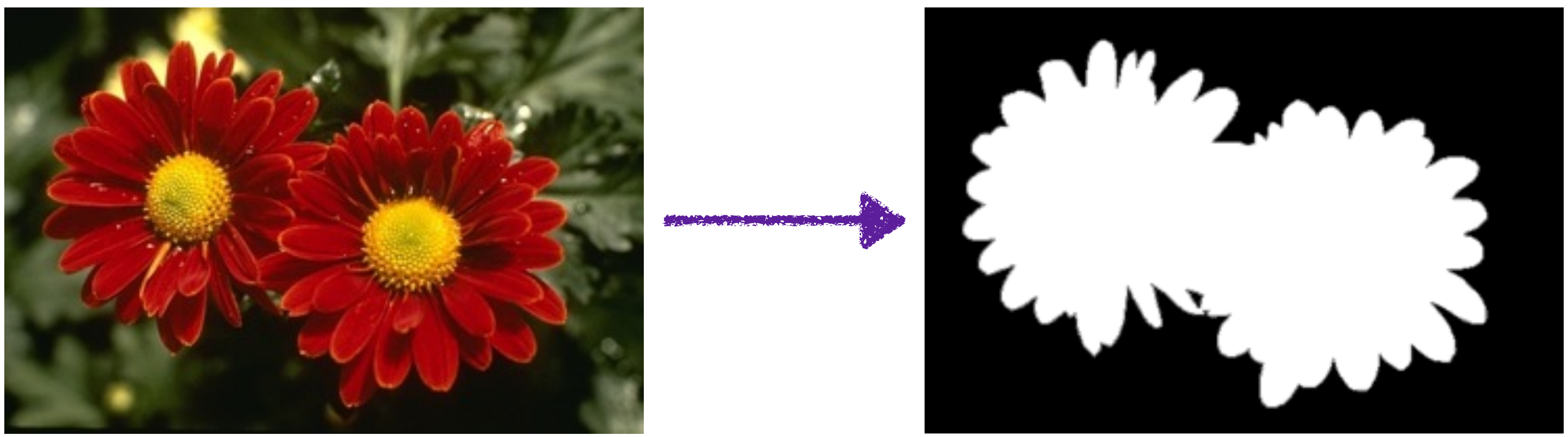
- How to choose your route to work?




- Choosing your route daily using perturb-max models is as good (on average) as choosing the best route [Kalai & Vempala 05, Cohen & Hazan 15]

# Loss minimization

- Can we learn from a finite sample set and generalize?



- Minimizing the average loss using perturb-max models generalize well [Hazan et al. 13]

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# Thank you