Performance analysis of stochastic networks using renormalization techniques

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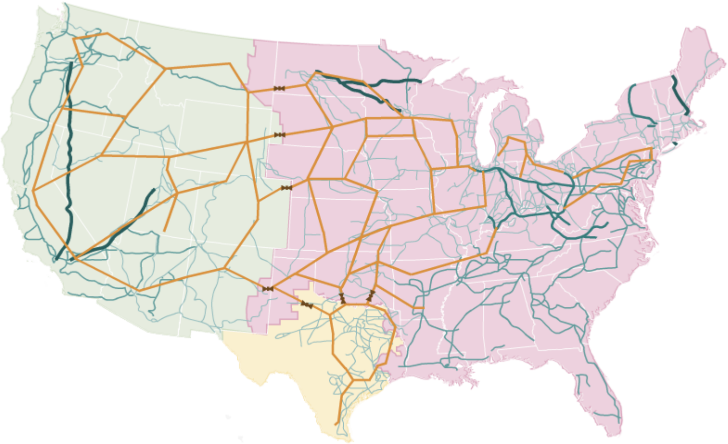
Performance analysis of stochastic networks

Renormalization techniques

Examples

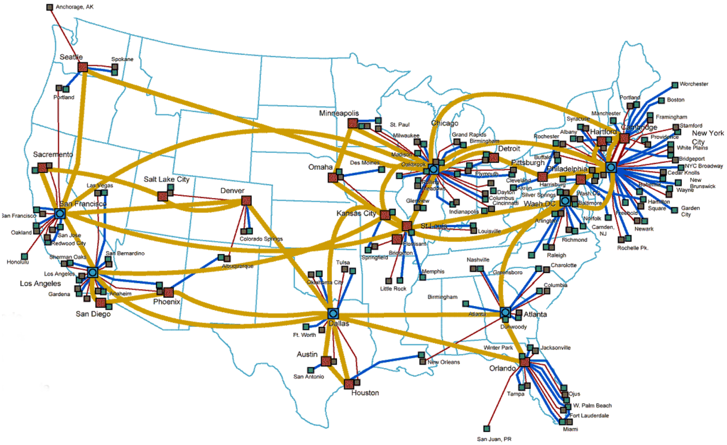
Stochastic networks

Resource sharing with uncertainty



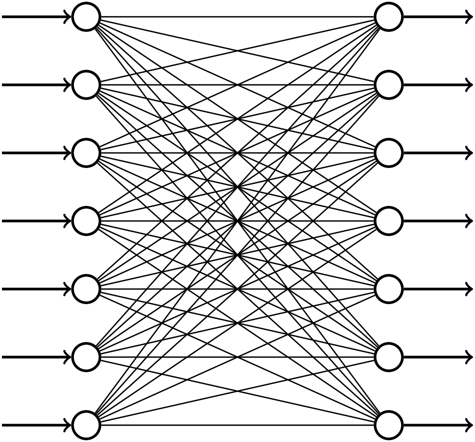
US electrical grid

European railroad network

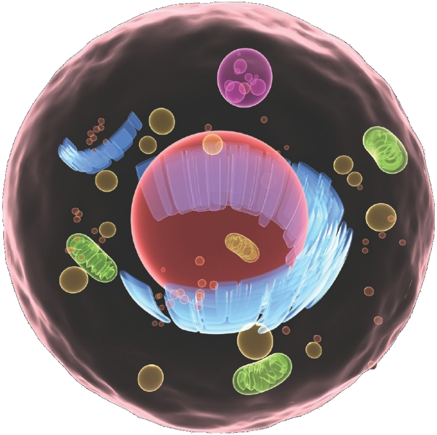


AT&T Internet backbone

Internet router



Internet router



Cell

Randomness

Several sources of randomness

- **Users behavior**
- **Failures**
- **Varying environment**
- **Movements**
- ...

Human networks

Designed and managed by humans

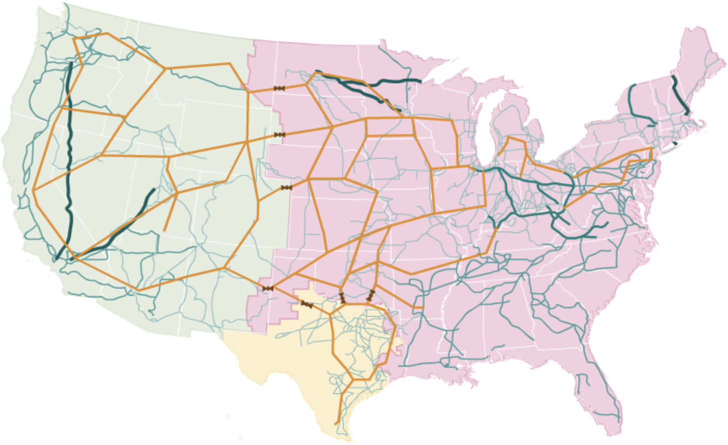
- Room for efficient design and management

1. Dimensioning:

- Known demand
- Target quality of service
- What is the network size?

2. Management:

- Network structure given
- More efficient use of resources

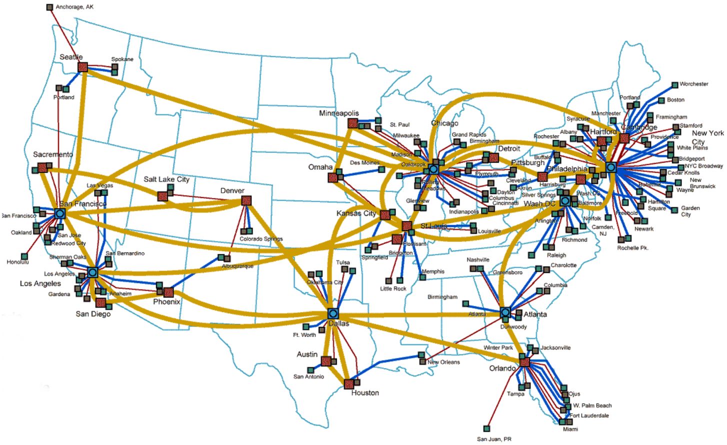


US electrical grid

- When to turn on/off power plants?
- Where to store energy?

European railroad network

- **How many trains to allocate?**
- **How to decide schedule?**



AT&T Internet backbone

- How to share links?
- How to route packets?

Internet router

- **Which packets to transmit?**

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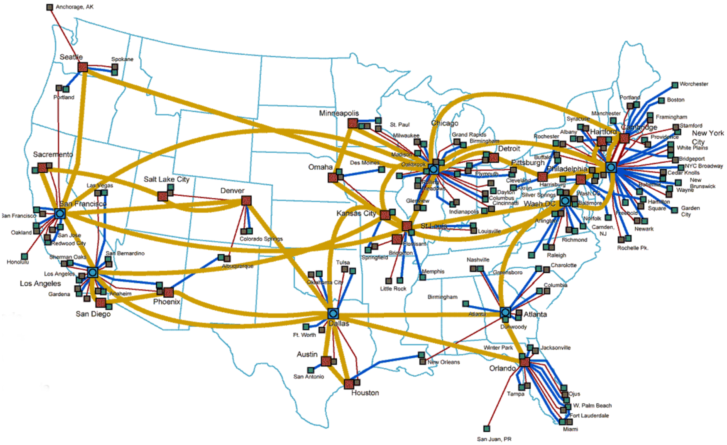


AT&T Internet backbone

- 2 million customers, 100 MB per day
- Which bandwidth so that delay ≤ 10ms?

Human networks

Designed and managed by humans

- Room for efficient design and management

1. Dimensioning:

- Known demand
- Target quality of service
- What is the network size?

2. Management:

- Network structure given
- More efficient use of resources

Measure of efficiency

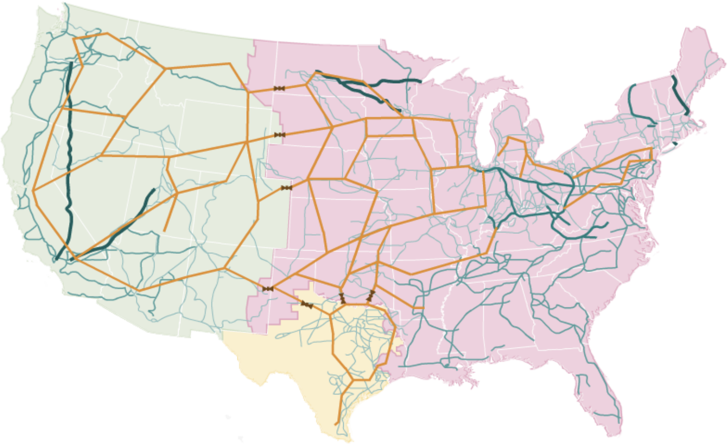
Performance analysis

Mathematical tools to assess efficiency of resource sharing algorithm

Two steps

1. Modeling
2. Mathematical analysis

Performance metric(s)?

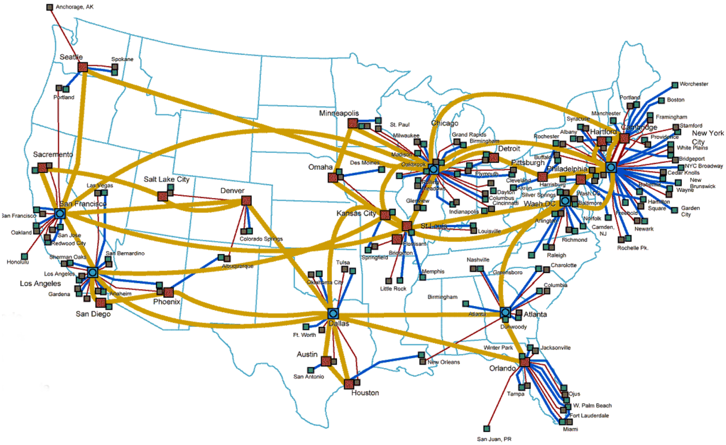


US electrical grid

- Frequency of black-outs
- Energy wasted

European railroad network

- **Delay**
- **Utilization**

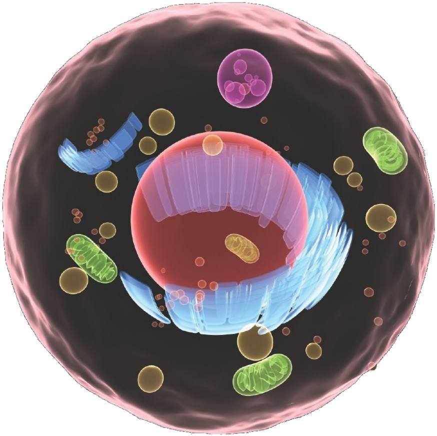


AT&T Internet backbone

- Latency/Jitter
- Throughput
- Fairness

Internet router

- **Throughput**
- **Delay**



Cell

- **Variability of protein expression**

Renormalization techniques

LLN and CLT

$(X_n, n \geq 1)$ **sequence of i.i.d. random variables**

Law of large numbers (LLN)

If $\mathbb{E}|X| < +\infty$**:**

$$\frac{1}{n}\sum_{k=1}^{n} X_k \underset{n\to+\infty}{\longrightarrow} \mathbb{E}X \quad \text{(almost surely)}$$

Central limit theorem (CLT)

If $\mathbb{E}X = 0$ **and** $\mathbb{E}X^2 = 1$**:**

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n} X_k \underset{n\to+\infty}{\Longrightarrow} \mathcal{N} \quad \text{(in distribution)}$$

LLN and CLT

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➼ Remark that $n^{-1/2}\sum_{k=1}^{n} X_k \Rightarrow 0$ by LLN

Functional LLN

Recall LLN:

- $(X_n, n \geq 1)$ **sequence of i.i.d. random variables**

Define new sequence $(S_n, n \geq 1)$ of random variables:

$$S_n = \frac{1}{n}(X_1 + \cdots + X_n)$$

Then $S_n \to \mathbb{E}X$

Functional LLN

Define new sequence $(\overline{S}_n, n \geq 1)$ of **processes**:

$$\overline{S}_n(t) = \frac{1}{n}\big(X_1 + \cdots + X_{\lfloor nt \rfloor}\big),\ t \geq 0$$

- **Speed up time by n, renormalize in space by n**

LLN: $\overline{S}_n(t)$ converges for each **fixed** t:

$$\overline{S}_n(t) = t \times \frac{1}{nt}\big(X_1 + \cdots + X_{\lfloor nt \rfloor}\big) \to t \times \mathbb{E}X$$

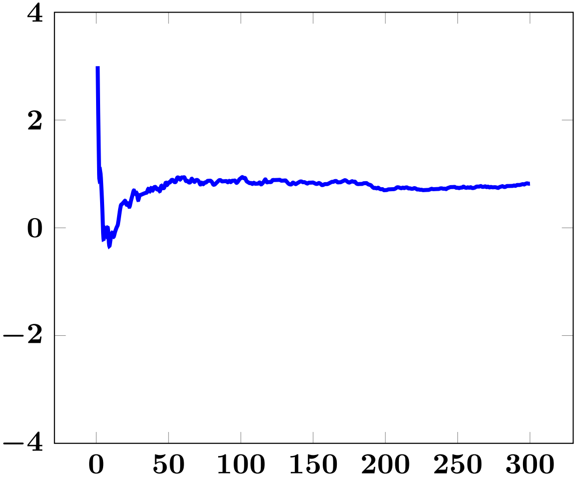
Functional LLN

Functional LLN: $\overline{S}_n \to \overline{S}_\infty = (t\mathbb{E}X, t \geq 0)$

- **Convergence of processes**
- **Uniform convergence on compact sets:**

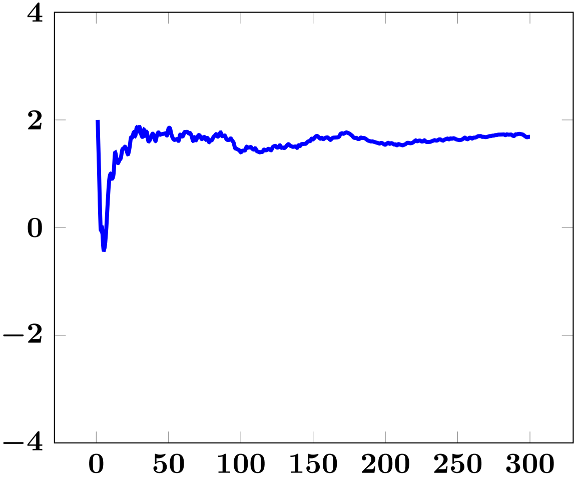
$$\sup_{0 \leq s \leq t} \left| \overline{S}_n(s) - \overline{S}_\infty(s) \right| \underset{n \to +\infty}{\longrightarrow} 0, \ t \geq 0$$

Functional LLN



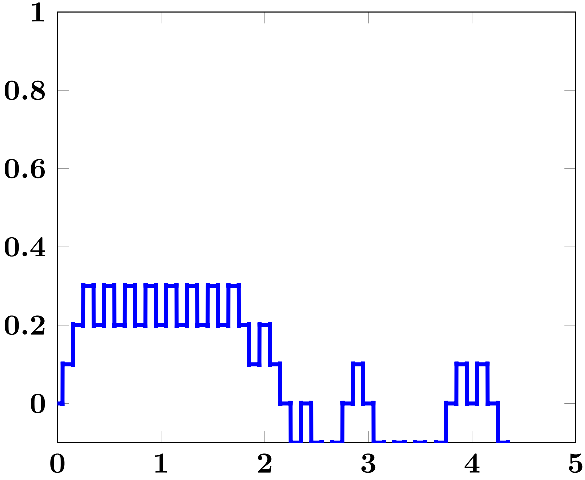
$\overline{S}_n(5)$ vs n

Functional LLN



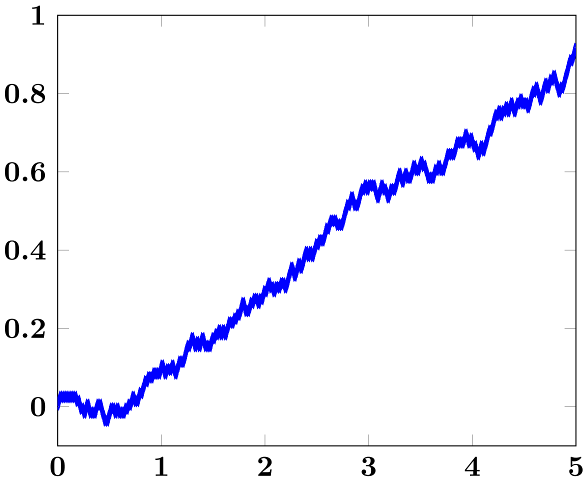
$\overline{S}_n(10)$ vs n

Functional LLN



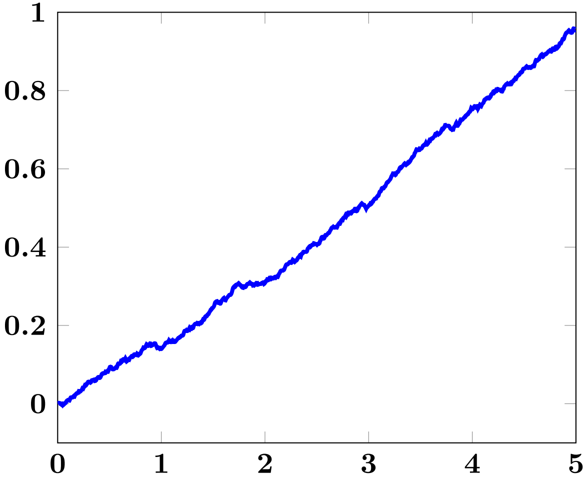
$\overline{S}_{10}(t)$ vs t

Functional LLN



$\overline{S}_{100}(t)$ vs t

Functional LLN



$\overline{S}_{1000}(t)$ vs t

Functional LLN

$$\boxed{\overline{S}_n \to \overline{S}_\infty}$$

$\mathbb{E}X = 0$: $\overline{S}_\infty = 0$

Functional CLT

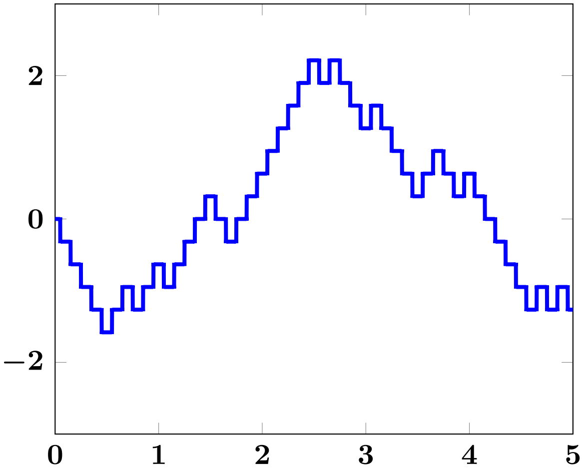
Define new sequence $(\widehat{S}_n, n \geq 1)$ of **processes:**

$$\widehat{S}_n(t) = \frac{1}{\sqrt{n}}\big(X_1 + \cdots + X_{\lfloor nt \rfloor}\big),\ t \geq 0$$

- **Speed up time by n, renormalize in space by $\sqrt{n}$**
- **CLT: $\widehat{S}_n(t) \Rightarrow \mathcal{N}$**

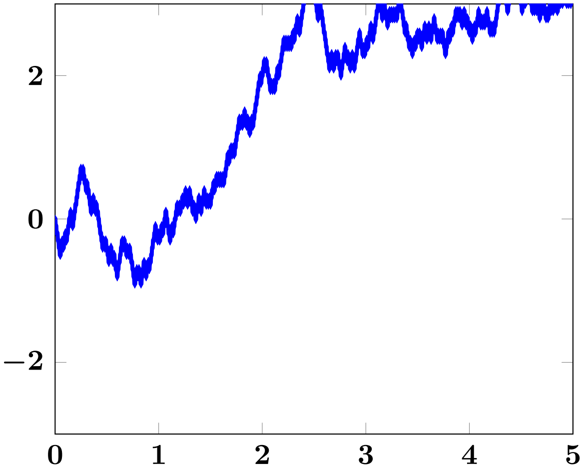
Functional CLT: $\widehat{S}_n \Rightarrow \widehat{S}_\infty$: Brownian motion

Functional CLT



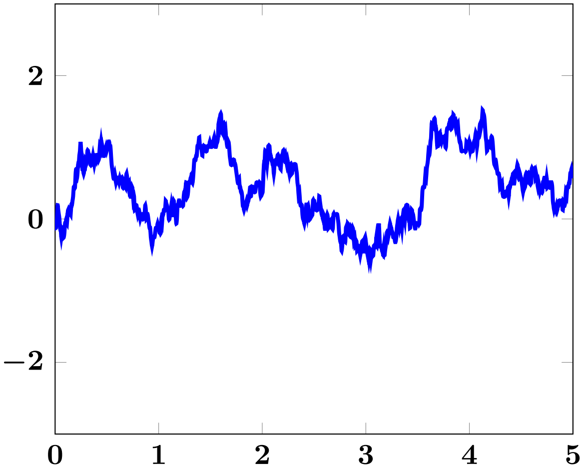
$\widehat{S}_{10}(t)$ vs t

Functional CLT



$\widehat{S}_{100}(t)$ vs t

Functional CLT



$\widehat{S}_{1000}(t)$ vs t

Added value 1/5

First and second order asymptotic expansions

- $\overline{S}_\infty$: mean behavior
- $\widehat{S}_\infty$: fluctuations around the mean

Added value 2/5

$\overline{S}_\infty$ and $\widehat{S}_\infty$ more tractable

- $\overline{S}_\infty$ is deterministic
- Can do computations on $\widehat{S}_\infty$:

$$\mathbb{P}\left(\sup_{s\leq t}\widehat{S}_\infty(s)\geq x\right)=\frac{\sqrt{2}}{\sqrt{\pi}}\int_x^{+\infty}e^{-y^2/2}\mathrm{d}y$$

whereas law of $\sup_{s\leq t}\widehat{S}_n(s)$ unknown

Added value 3/5

Convergence of all continuous functionals:

$$\boxed{\Psi(\overline{S}_n) \to \Psi(\overline{S}_\infty) \quad \text{and} \quad \Psi(\widehat{S}_n) \Rightarrow \Psi(\widehat{S}_\infty)}$$

- **Interest of uniform convergence**

Added value 4/5

Invariance principle:

$$\boxed{\overline{S}_\infty = f(\mathbb{E}X) \quad \text{and} \quad \widehat{S}_\infty = f(\mathbb{E}X, \mathbb{E}X^2)}$$

$\overline{S}_\infty$ **and** $\widehat{S}_\infty$ **the same for any other** Y **with** $\mathbb{E}Y^{1,2} = \mathbb{E}X^{1,2}$

Get to know the "true" parameters

- **Dimensioning**

Added value 5/5 (more technical)

Stability (positive recurrence) of Markov processes:

- **Difficult issue**
- **Important performance metric**

Added value 5/5 (more technical)

Set-up

- $(M(t), t \geq 0)$: Markov process, countable state-space
- Sequence (m_n) of initial states of size $\|m_n\| = n$
- Renormalize by size of initial state (LLN scaling):

$$\overline{M}_n(t) = \frac{1}{n} M(nt) \quad \text{when} \quad M(0) = m_n$$

Theorem (informal)

If for every sequence (m_n) as above, $\overline{M}_n \to \overline{M}_\infty$ and $\overline{M}_\infty(t) = 0$ for all t large enough, then M is stable.

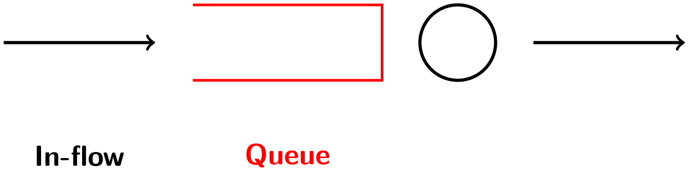
Examples

Single server queue

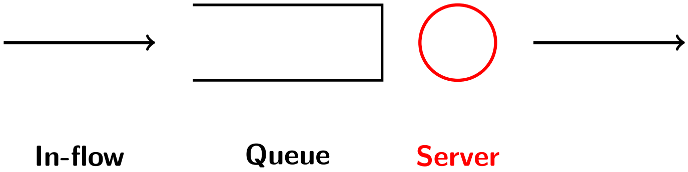
Single server queue

In-flow

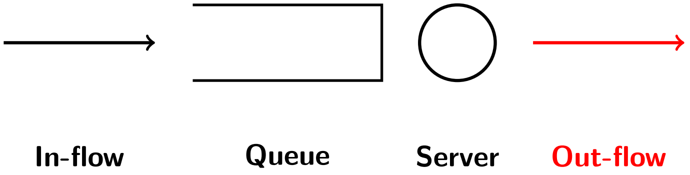
Single server queue



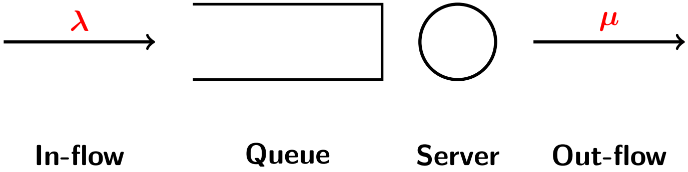
Single server queue



Single server queue



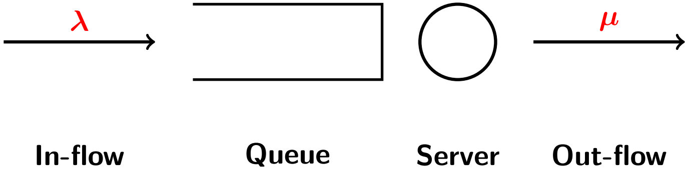
Single server queue



- Arrivals at rate λ
- Service capacity μ

Stability condition: $\lambda < \mu$

Single server queue



$Q(t) = \#$ customers in queue at time t

- Exponential assumptions: Q Markov process

$$Q(t) \longrightarrow \begin{cases} Q(t)+1 & \text{at rate } \lambda \\ Q(t)-1 & \text{at rate } \mu \text{ if } Q(t) > 0 \end{cases}$$

- Random walk reflected at 0

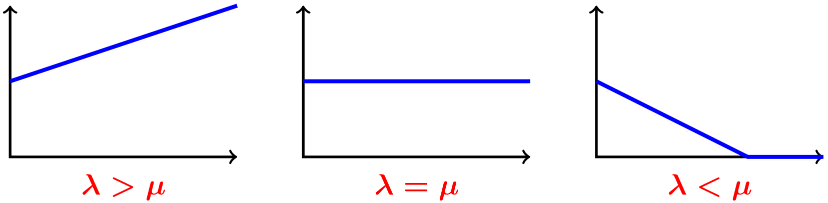
Single server queue: LLN

Renormalization:

$$\overline{Q}_n(t) = \frac{1}{n} Q_n(nt) \quad \text{with} \quad Q_n(0) = n$$

Functional LLN: $\overline{Q}_n \to \overline{Q}_\infty(t) = 1 + (\lambda - \mu t)^+$

- $q^+ = \max(q, 0)$: reflection at 0
- First-order behavior: increase at rate λ, decrease at rate μ



Single-server queue: CLT

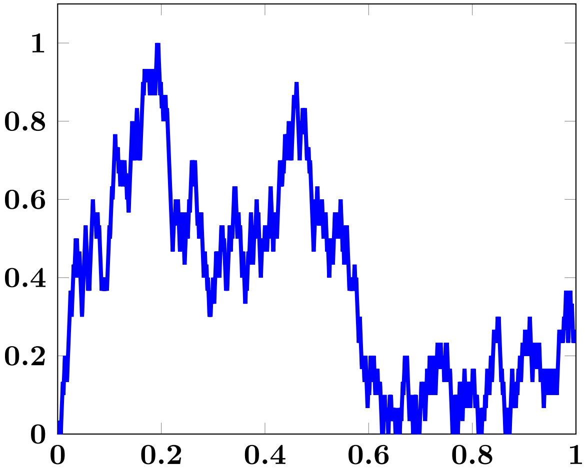
Renormalization when $\lambda = \mu$:

$$\widehat{Q}_n(t) = \frac{1}{\sqrt{n}} Q(nt)$$

Functional CLT: $\widehat{Q}_n \Rightarrow \widehat{Q}_\infty$

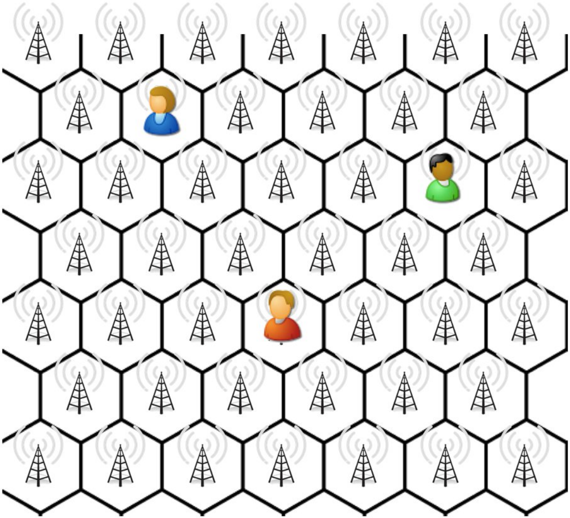
- $\widehat{Q}_\infty$: reflected Brownian motion

Single-server queue: CLT



$\widehat{Q}_\infty(t)$ vs t

Wireless network with mobile users



Bandwidth allocation?

Markovian model

Network with K nodes labelled $k = 1, \ldots, K$

Node k:

- Arrival rate λ_k
- Service capacity μ_k

Customers:

- Require i.i.d. $\exp(1)$ service requirements
- Move independently: common kernel $R = (r_{k\ell})$

$r_{k\ell}$ = rate at which **each** user moves from k to ℓ

Markovian model

$N_k(t)$ = # users at node k at time t

$N(t) = (N_1(t), \dots, N_K(t))$:

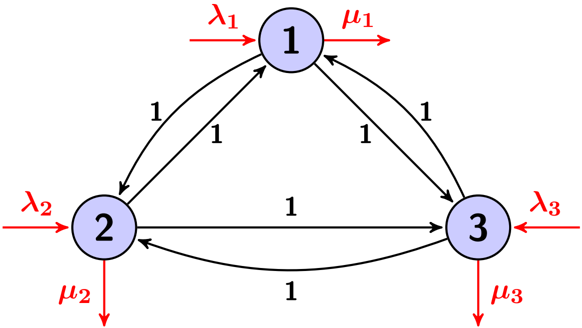
- K-dimensional Markov process
- Transition rates:

$$n \longrightarrow \begin{cases} n + e_k & \text{at rate } \lambda_k & \text{(arrival)} \\ n - e_k & \text{at rate } \mu_k \mathbb{1}_{\{n_k > 0\}} & \text{(departure)} \\ n - e_k + e_\ell & \text{at rate } n_k \times r_{k\ell} & \text{(movement)} \end{cases}$$

  e_k = kth unit vector $(0, \dots, 0, 1, 0, \dots, 0)$
- $Q(t) = N_1(t) + \cdots + N_K(t)$: total # customers

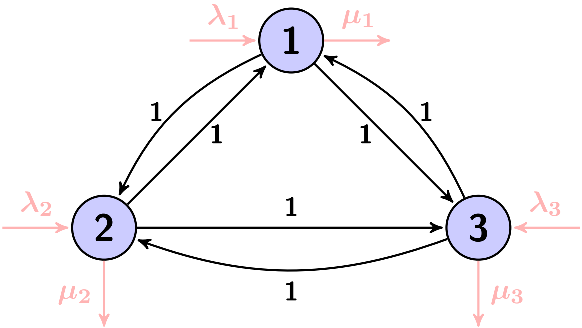
Symmetric example

Consider the case $K = 3$ and $r_{k\ell} = 1$



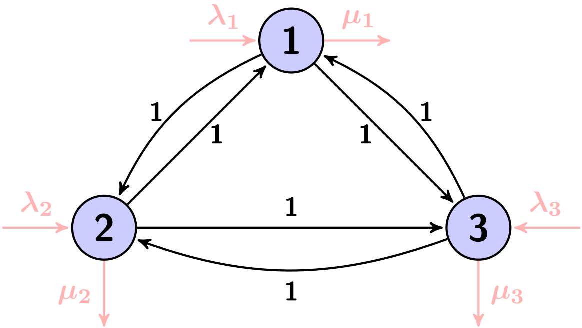
Symmetric example

Focus on movements



Symmetric example

Focus on movements



Each customer in cell k with probability $1/3$

LLN: $N_k(t) \approx \frac{1}{3}Q(t)$ when $Q(t) \gg 1$

Dimension reduction

Assume R irreducible, stationary distribution π

- π_k: probability of being at node k

When $Q(t) \gg 1$: $N_k(t) \approx Q(t) \times \pi_k$ for each k

$$N(t) \approx Q(t) \times \pi$$

K-dimensional

K-dimensional

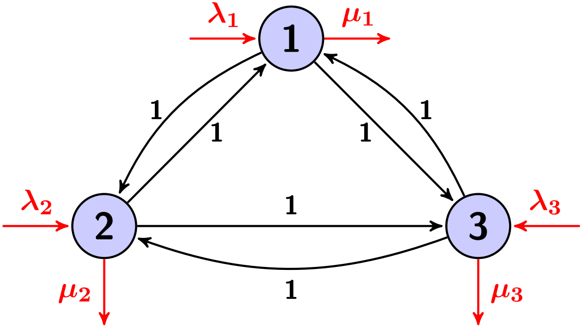
1-dimensional

Reduces the problem to the study of $Q(t)$

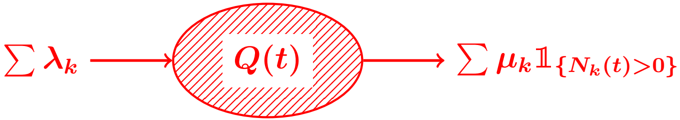
- 1-dimensional problem!

Single-server queue analogy

Focus on arrivals/departures

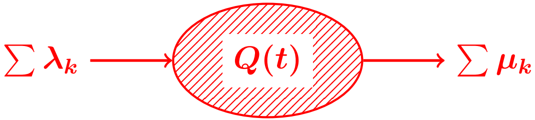


Single-server queue analogy



$\mathbb{1}_{\{N_k(t)>0\}} \rightarrow$ **non-Markovian**

Single-server queue analogy



No empty node $\rightarrow$ departure rate $= \sum \mu_k$

Whole system behaves as single-server queue

- **Arrival rate $\lambda = \sum \lambda_k$**
- **Service capacity $\mu = \sum \mu_k$**

$\mathbb{P}$(**no empty node**) ≈ 1 **when** $Q(t) \gg 1$

Functional LLN

Theorem (with D. Tibi, Annals of Applied Probability 2010)
Let $\overline{N}_n(t) = N^n(nt)/n$ with $|N^n(0)| = n$: then

$$(\overline{N}_n(t), t \geq 0) \underset{n\to+\infty}{\Longrightarrow} \overline{Q}_\infty \times \pi$$

- $\overline{Q}_\infty(t) = 1 + (\lambda - \mu)^+ t$: limit of single-server queue

Consequence

- Stability condition: $\sum \lambda_k \leq \sum \mu_k$

Functional CLT

Theorem (with S. Borst, Queueing Systems 2013)

Assume that $\sum \lambda_k = \sum \mu_k$ and let $\overline{N}_n(t) = N(n^2 t)/n$: then

$$(\widehat{N}_n(t), t \geq 0) \underset{n \to +\infty}{\Longrightarrow} \widehat{Q}_\infty \times \pi$$

- $\widehat{Q}_\infty$: reflected Brownian motion, limit of single-server queue

Bandwidth-sharing

Extensions

- **Multiclass, α-fair bandwidth-sharing**