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Région PACA

Vendredi 17 mars

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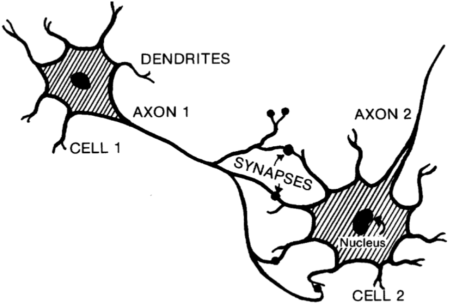
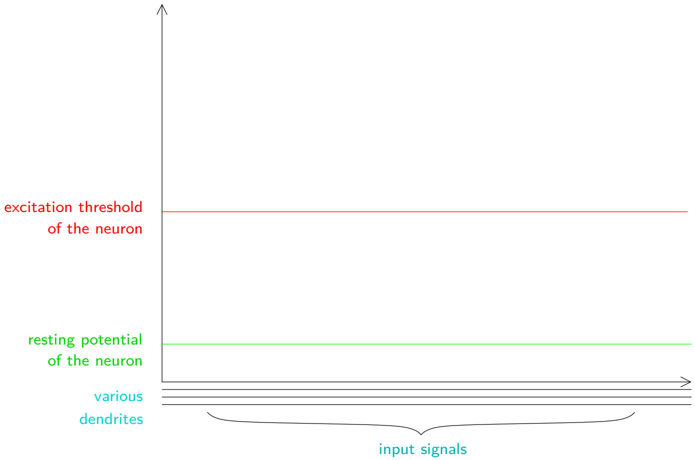
Transmission of the neuronal information

The neuronal information is transmitted by the spikes/action potentials.



Introduction of the biological context

Synaptic integration



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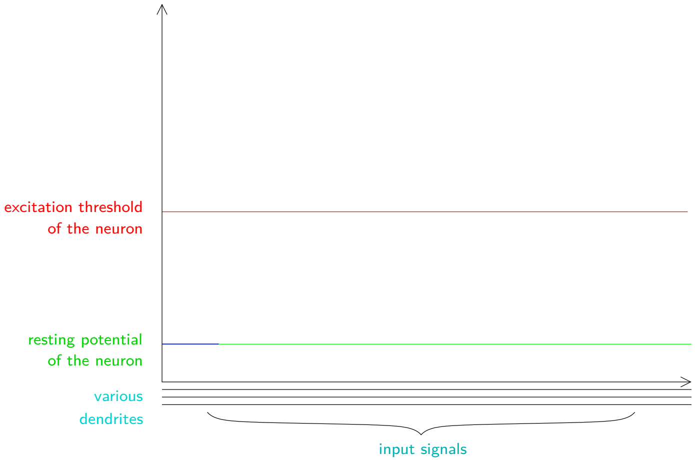

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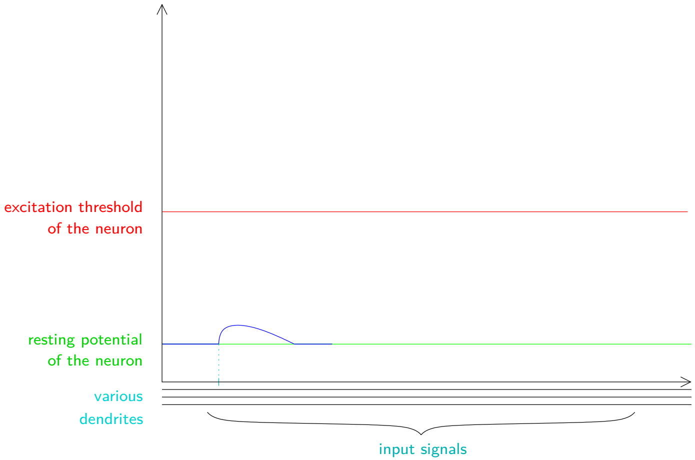

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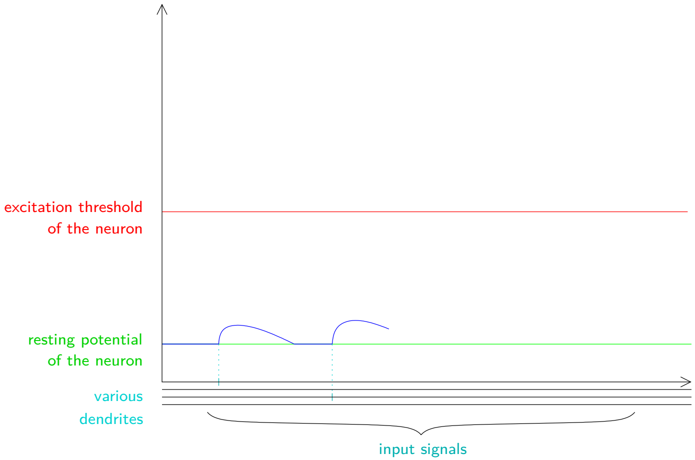

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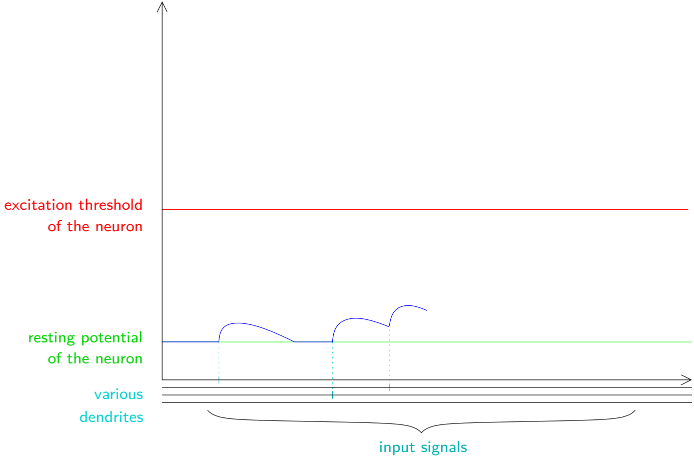

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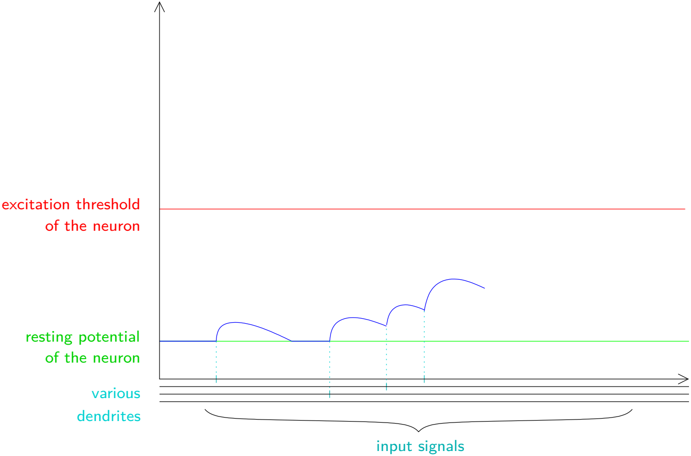

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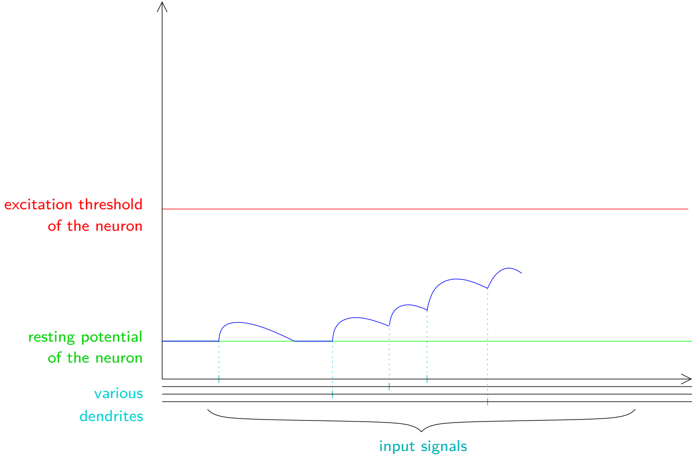

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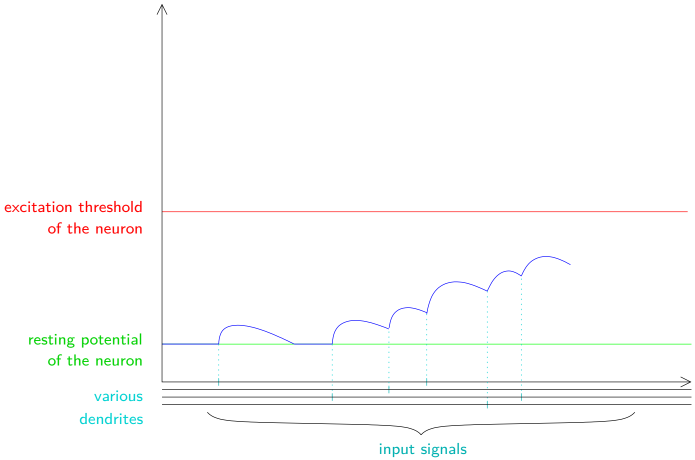

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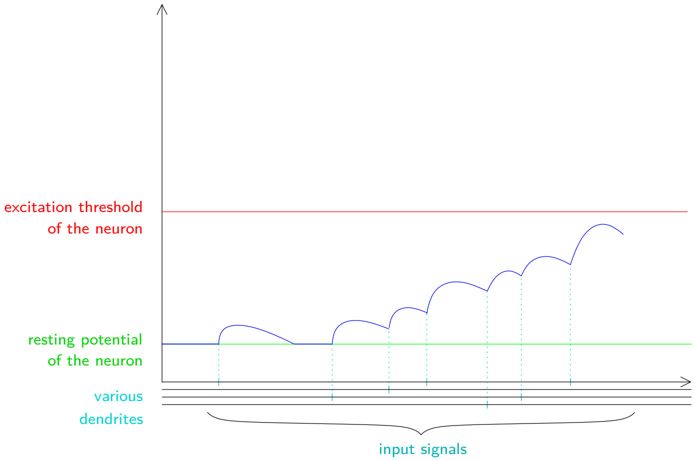

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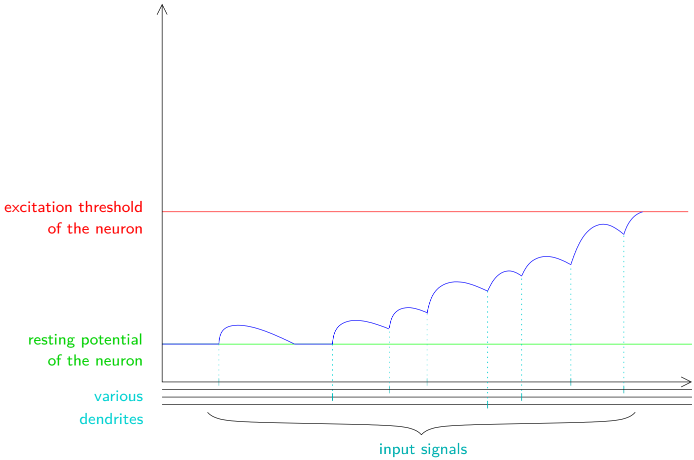

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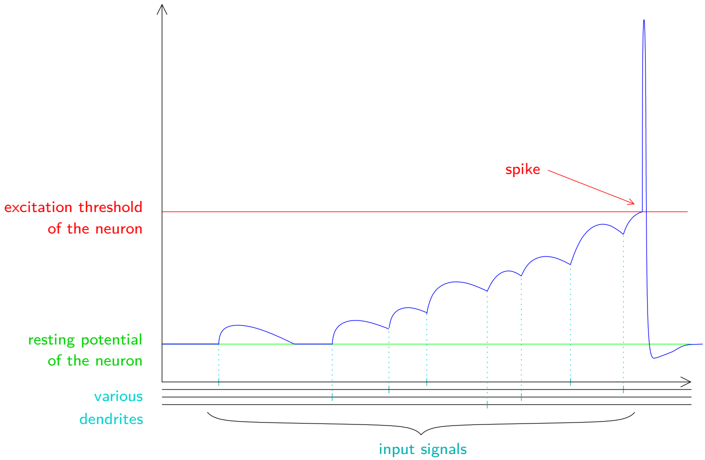

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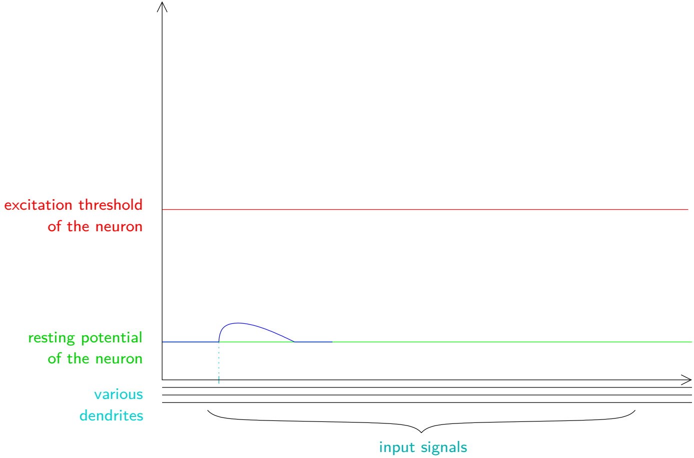

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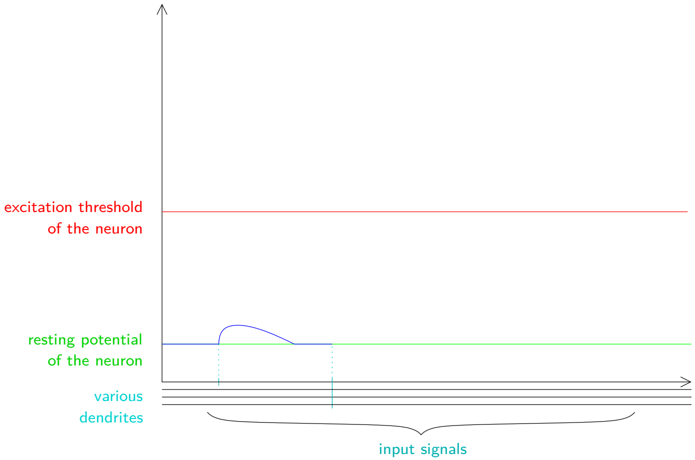

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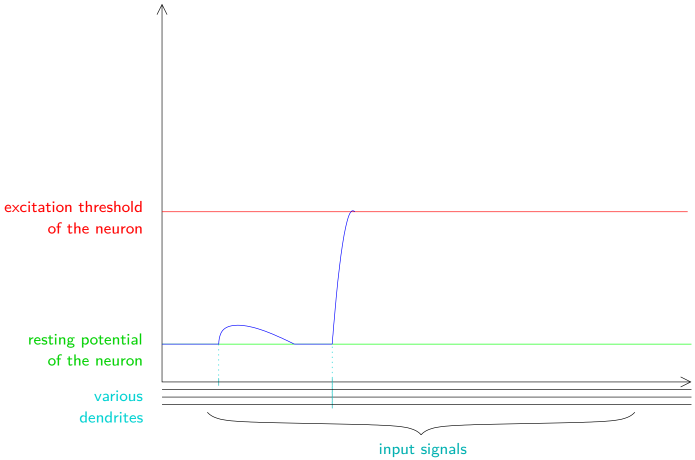

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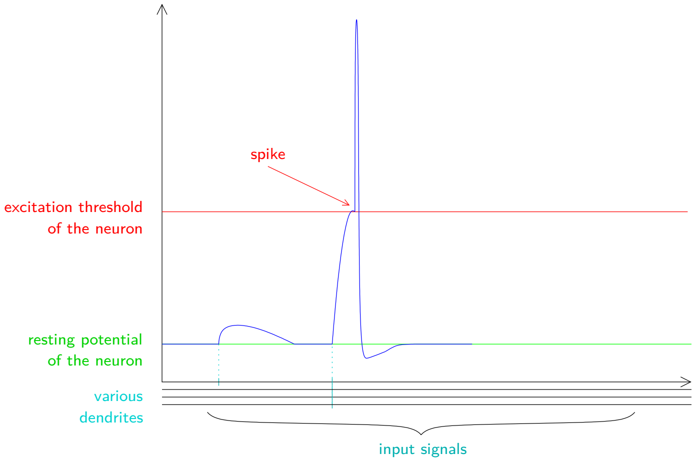

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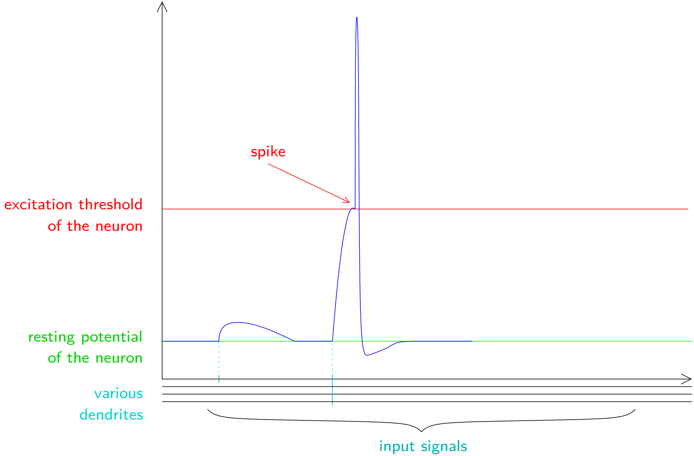
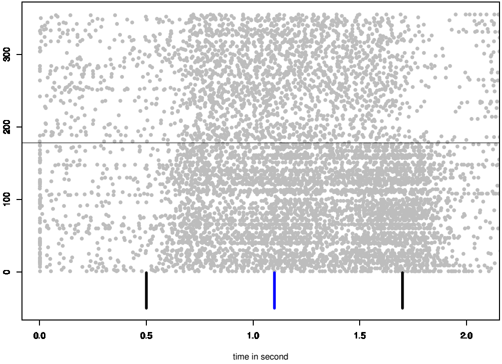


[Grammont and Riehle (1999)]: neurons coordinate their activity at very precise moments.

Introduction of the biological context

Synchronization detection



Observation

Raster plot

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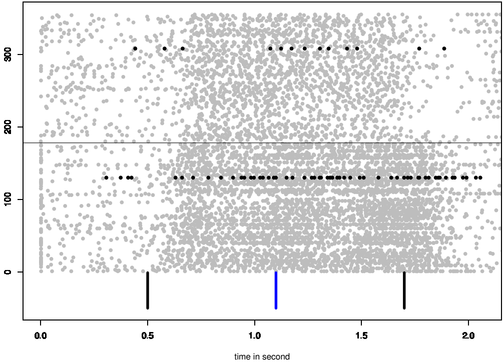
Synchronization detection



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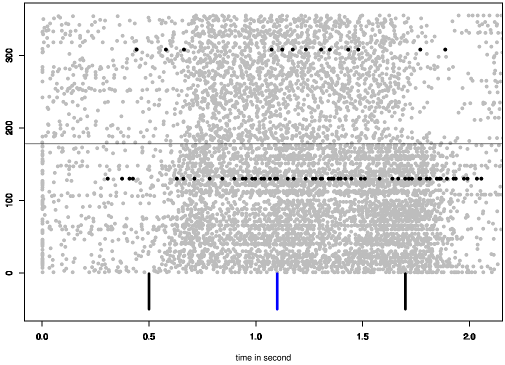


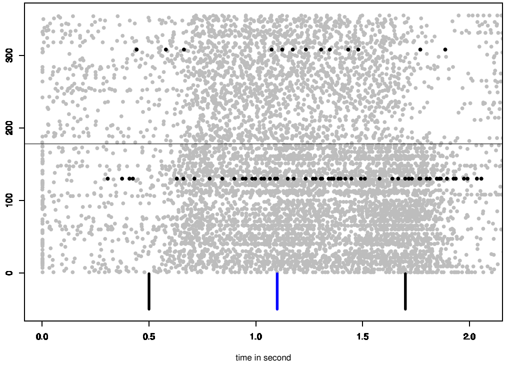
Observation

Raster plot

Modelling

Point processes
= random sets of points

⚠ Complex dependences



Synchronization

Tendency to
co-activate

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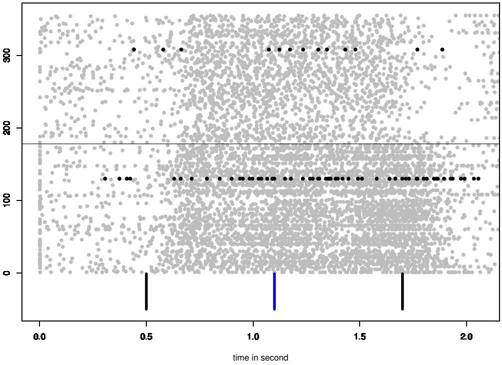

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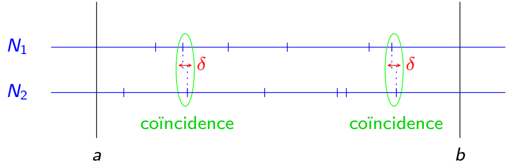
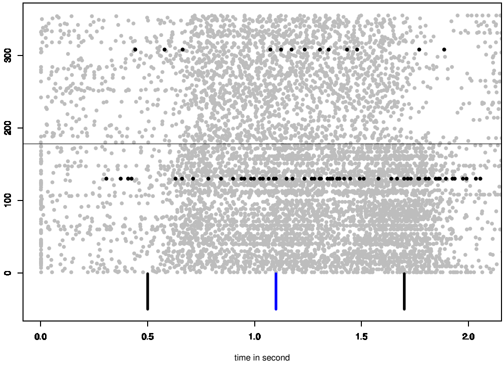
Synchronization

Tendency to co-activate

Coincidence





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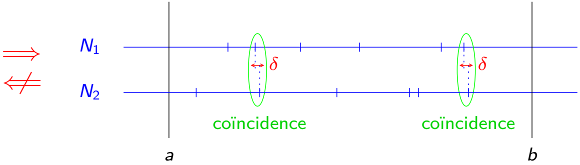

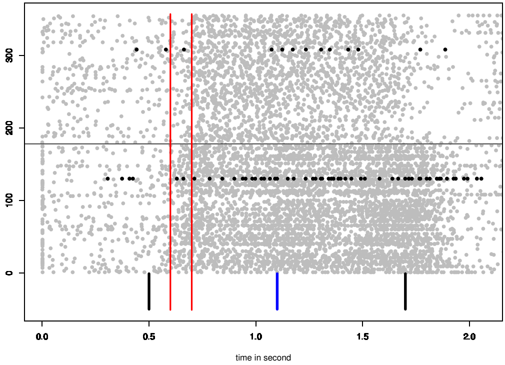

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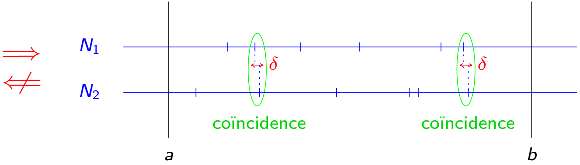

Introduction of the biological context

Global approach

- On each time window:

Observed total number of coincidences	VS	Expected under independence total number of coincidences

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Global approach

- On each time window:

Observed total number of coincidences $\ggg$ Expected under independence total number of coincidences

$\Longrightarrow$ occurence of a synchronization.

Global approach

- On each time window: independence test.

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Global approach

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- Simultaneous application on each tim window: multiple tests.

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State of the art in neurosciences

Parametric methods
(eg: Poisson point processes)

$\rightarrow$ not realistic in neuroscience.

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State of the art in neurosciences

Parametric methods
(eg: Poisson point processes)

$\rightarrow$ not realistic in neuroscience.

Non-parametric approaches
(eg: *trial shuffling*)

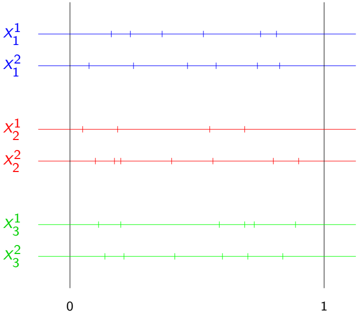
$\rightarrow$ not theoretically justified.

Statistical framework

Observation:

$$\mathbb{X}_n = (X_1, \dots, X_n), \quad \text{where } X_i = (X_i^1, X_i^2) \text{ i.i.d.}$$

Aim at testing $(\mathcal{H}_0): \; X^1 \perp\!\!\!\perp X^2$ against $(\mathcal{H}_1): \; X^1 \not\perp\!\!\!\perp X^2$.



$\mathbb{X}_n$: observed sample

$\{X_1, X_2, X_3, ...\}$ i.i.d.

Statistical framework

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$$\mathbb{X}_n = (X_1, \ldots, X_n), \quad \text{where } X_i = (X_i^1, X_i^2) \text{ i.i.d.}$$

Aim at testing $(\mathcal{H}_0) : \; X^1 \perp\!\!\!\perp X^2$ against $(\mathcal{H}_1) : \; X^1 \not\perp\!\!\!\perp X^2$.

Often, tests based on $\iint \varphi(x^1, x^2) \left(dP(x^1, x^2) - dP^1(x^1) dP^2(x^2) \right)$.

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State of the art:

$$n^{-3/2} \sup_{v^1 \in \mathcal{V}^1, v^2 \in \mathcal{V}^2} \left| \sum_{i \neq i'} \left(\varphi_{(v^1,v^2)} \left(X_i^1, X_i^2 \right) - \varphi_{(v^1,v^2)} \left(X_i^1, X_{i'}^2 \right) \right) \right|,$$

with

- [Blum, Kiefer, and Rosenblatt (1961)], for $\mathcal{V}^1 = \mathcal{V}^2 = \mathbb{R}$ and $\varphi_{(v^1,v^2)}(x^1, x^2) = \mathbb{1}_{\{]-\infty, v^1]\}}(x^1) \times \mathbb{1}_{\{]-\infty, v^2]\}}(x^2)$,
- [Romano (1989)], bootstrap and permutation for $\mathcal{V}^1$ and $\mathcal{V}^2$ countable V.-C. classes of subsets, and $\varphi_{(v^1,v^2)}(x^1, x^2) = \mathbb{1}_{\{v^1\}}(x^1) \times \mathbb{1}_{\{v^2\}}(x^2)$,
- [Van der Vaart and Wellner (1996)], bootstrap for $\mathcal{V}^1$ and $\mathcal{V}^2$ classes of real-valued functions, and $\varphi_{(v^1,v^2)}(x^1, x^2) = v^1(x^1) \times v^2(x^2)$.

Statistical framework

Observation:

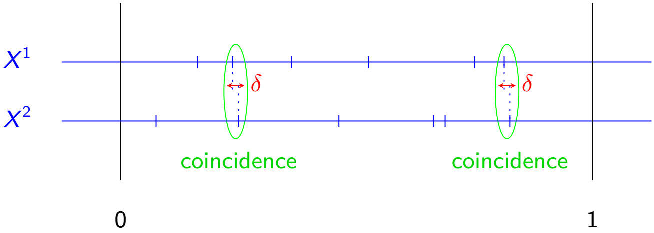
$$\mathbb{X}_n = (X_1, \ldots, X_n), \quad \text{where } X_i = (X_i^1, X_i^2) \text{ i.i.d.}$$

Aim at testing $(\mathcal{H}_0): \; X^1 \perp\!\!\!\perp X^2$ against $(\mathcal{H}_1): \; X^1 \not\!\perp\!\!\!\perp X^2$.

Notion of (delayed) coincidence [Tuleau-Malot et al. (2014)]

φ_δ^{coinc} counts the number of coincidences between two point processes:

$$\varphi_\delta^{coinc}\left(X^1, X^2\right) = \sum_{T \in X^1} \sum_{S \in X^2} \mathbb{1}_{\{|T-S| \leq \delta\}}.$$



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Unbiased estimator:

$$\frac{1}{n(n-1)} \sum_{i \neq i'} \left(\varphi\left(X_i^1, X_i^2\right) - \varphi\left(X_i^1, X_{i'}^2\right) \right).$$

Statistical framework

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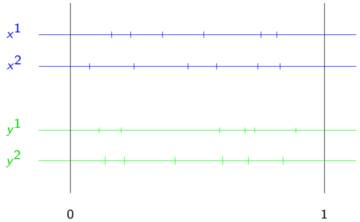
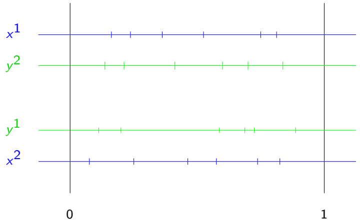
Unbiased estimator:

$$\frac{1}{n(n-1)} \sum_{i \neq i'} \left(\varphi\left(X_i^1, X_i^2\right) - \varphi\left(X_i^1, X_{i'}^2\right) \right).$$

Let $h_\varphi\left((x^1, x^2), (y^1, y^2)\right) = \frac{1}{2}\left(\varphi(x^1, x^2) + \varphi(y^1, y^2) - \varphi(x^1, y^2) - \varphi(y^1, x^2)\right)$,



$\varphi(x^1, x^2) + \varphi(y^1, y^2)$



$-\varphi(x^1, y^2) - \varphi(y^1, x^2)$

Statistical framework

Observation:

$$\mathbb{X}_n = (X_1, \dots, X_n), \quad \text{where } X_i = (X_i^1, X_i^2) \text{ i.i.d.}$$

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Unbiased estimator:

$$\frac{1}{n(n-1)} \sum_{i \neq i'} \left(\varphi\left(X_i^1, X_i^2\right) - \varphi\left(X_i^1, X_{i'}^2\right) \right).$$

Let $h_\varphi\left((x^1, x^2), (y^1, y^2)\right) = \frac{1}{2}\left(\varphi(x^1, x^2) + \varphi(y^1, y^2) - \varphi(x^1, y^2) - \varphi(y^1, x^2)\right)$,

the previous estimator becomes $\displaystyle \frac{1}{n(n-1)} \sum_{i \neq j} h_\varphi\left(X_i, X_j\right).$

Test statistic

$$\sqrt{n}U_{n,h}(\mathbb{X}_n) = \frac{\sqrt{n}}{n(n-1)} \sum_{i \neq j} h(X_i, X_j),$$

for $h : (\mathcal{X} \times \mathcal{X})^2 \to \mathbb{R}$ well chosen such that

$$(\mathcal{A}^{Cent}) \begin{cases} \text{for any } X \text{ and } X', \text{ i.i.d. with distribution } P^1 \otimes P^2 \text{ on } \mathcal{X}^2, \\ \mathbb{E}[h(X, X')] = 0, \end{cases}$$

and that $U_{n,h}$ is non-degenerate.

Independence test

Generalized test statistic

Test statistic

$$\sqrt{n}U_{n,h}\left(\mathbb{X}_n\right) = \frac{\sqrt{n}}{n(n-1)} \sum_{i \neq j} h\left(X_i, X_j\right),$$

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TCL for our test statistic under independence

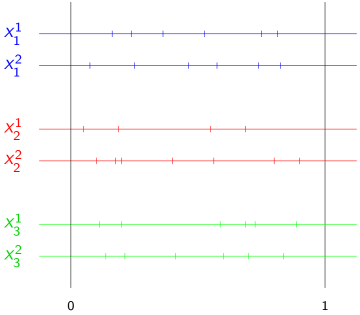
If $U_{n,h}$ is non-degenerate, then

$$\mathcal{L}\left(\sqrt{n}U_{n,h}; P^1 \otimes P^2\right) \underset{n \to +\infty}{\Longrightarrow} \mathcal{N}\left(0, \sigma^2_{\perp\!\!\!\perp}\right).$$

$\mathcal{L}\left(\sqrt{n}U_{n,h}; Q\right)$ denotes the distribution of $\sqrt{n}U_{n,h}\left(\mathbb{Z}_n\right)$ for $\mathbb{Z}_n = (Z_1, \ldots, Z_n)$ sample of i.i.d. random variables with distribution Q.

Bootstrap approach inspired by Romano (1988)

Given $\mathbb{X}_n = (X_i)_{1 \le i \le n}$, where $X_i = \left(X_i^1, X_i^2\right)$ i.i.d. $\sim P$ in $\mathcal{X} \times \mathcal{X}$.



$\sqrt{n} U_{n,h}(\mathbb{X}_n)$: on original data

Bootstrap approach

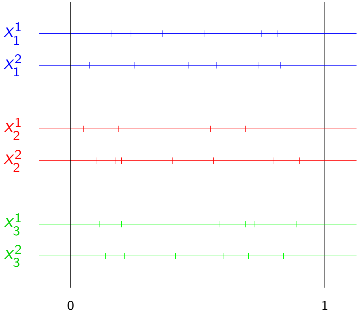
Description

Bootstrap approach inspired by Romano (1988)

Given $\mathbb{X}_n = (X_i)_{1\le i\le n}$, where $X_i = \left(X_i^1, X_i^2\right)$ i.i.d. $\sim P$ in $\mathcal{X}\times\mathcal{X}$.

The bootstrap sample is

$$\mathbb{X}_n^* = \left(X_{n,1}^*, \ldots, X_{n,n}^*\right) \text{ i.i.d. } \sim \; P_n^1 \otimes P_n^2 = \left(\frac{1}{n}\sum_{1\le i\le n}\delta_{X_i^1}\right)\otimes\left(\frac{1}{n}\sum_{1\le j\le n}\delta_{X_j^2}\right).$$



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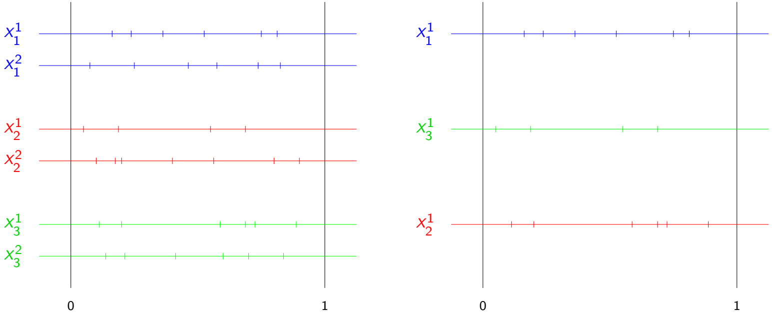

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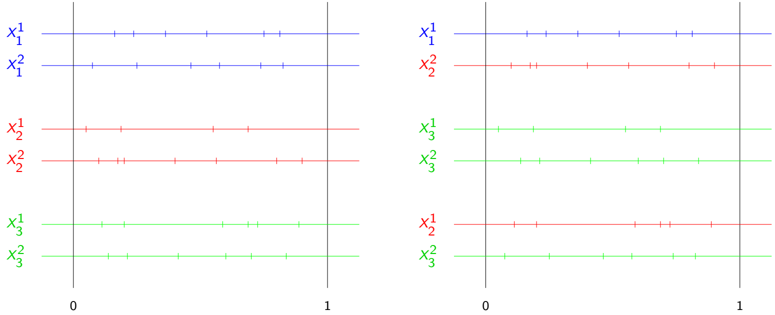
$$\mathbb{X}_n^* = \left(X_{n,1}^*, \ldots, X_{n,n}^*\right) \text{ i.i.d. } \sim P_n^1 \otimes P_n^2 = \left(\frac{1}{n}\sum_{1 \le i \le n} \delta_{X_i^1}\right) \otimes \left(\frac{1}{n}\sum_{1 \le j \le n} \delta_{X_j^2}\right).$$



$\sqrt{n}U_{n,h}(\mathbb{X}_n)$: on original data $\quad$ $\sqrt{n}U_{n,h}(\mathbb{X}_n^*)$: on bootstrapped data

Bootstrap approach

Consistency of the method

Theorem 1

Let $h : (\mathcal{X} \times \mathcal{X})^2 \to \mathbb{R}$ such that $(\mathcal{A}^{Cent})$, $(\mathcal{A}^{Cent}_{Emp})$

$$(\mathcal{A}^{Cent}_{Emp}) \left\{ \begin{array}{l} \forall x_1 = (x_1^1, x_1^2), \ldots, x_n = (x_n^1, x_n^2) \in \mathcal{X}^2, \\ \displaystyle\sum_{i,i',j,j'=1}^{n} h\left(\left(x_i^1, x_{i'}^2\right), \left(x_j^1, x_{j'}^2\right)\right) = 0. \end{array} \right.$$

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$$(\mathcal{A}^{Mmt}_{brk}) \begin{cases} \text{for any } X_1, X_2, X_3, X_4, \text{ i.i.d. with distribution } P \text{ on } \mathcal{X}^2, \\ \forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \{1, 2, 3, 4\}, \\ \mathbb{E}\left[h^2\left(\left(X^1_{\mathfrak{a}}, X^2_{\mathfrak{b}}\right), \left(X^1_{\mathfrak{c}}, X^2_{\mathfrak{d}}\right)\right)\right] < +\infty. \end{cases}$$

Theorem 1

Let $h : (\mathcal{X} \times \mathcal{X})^2 \to \mathbb{R}$ such that $(\mathcal{A}^{Cent})$, $(\mathcal{A}^{Cent}_{Emp})$, $(\mathcal{A}^{Mmt}_{brk})$ and $(\mathcal{A}^{Cont})$

$(\mathcal{A}^{Cont})$ $\begin{cases} \text{There exists } \mathcal{C} \text{ subset of } \mathcal{X}^2 \times \mathcal{X}^2, \text{ such that} \\ \text{(i) the kernel } h \text{ is continuous in every } (x, y) \in \mathcal{C} \\ \quad \text{for the topology induced by the Skorohod metric,} \\ \text{(ii) } (P^1 \otimes P^2)^{\otimes 2}(\mathcal{C}) = 1. \end{cases}$

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$$d_{W_2}\left(\mathcal{L}\left(\sqrt{n}U_{n,h}; P_n^1 \otimes P_n^2 | \mathbb{X}_n\right), \mathcal{L}\left(\sqrt{n}U_{n,h}; P^1 \otimes P^2\right)\right) \underset{n \to +\infty}{\longrightarrow} 0$$

P-a.s. in $(X_n)_n$.

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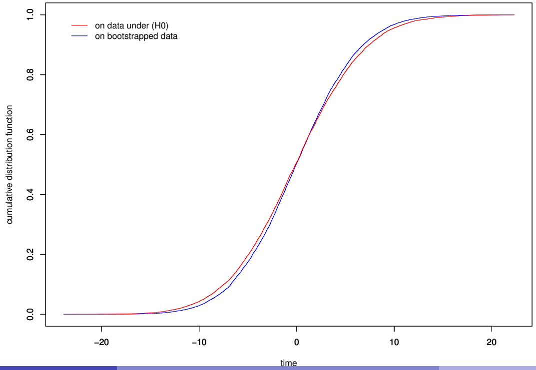

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Main arguments:

- By Varadarajan [1958],

$$\text{Separability} \quad \Longrightarrow \quad P\text{-a.s, } P_n^1 \otimes P_n^2 \underset{n\to+\infty}{\Longrightarrow} P^1 \otimes P^2.$$

- Skorohod's representation theorem.
- Continuity of h and LLN for U-statistics.

Bootstrap test of independence

$$\Phi^{+}_{h,\alpha} = \mathbb{1}_{\left\{\sqrt{n}U_{n,h}(\mathbb{X}_n) > q^*_{h,1-\alpha}(\mathbb{X}_n)\right\}},$$

where $q^*_{h,1-\alpha}(\mathbb{X}_n)$ is the (1-α)-quantile of $\mathcal{L}\left(\sqrt{n}U_{n,h}; P^1_n \otimes P^2_n|\mathbb{X}_n\right)$.

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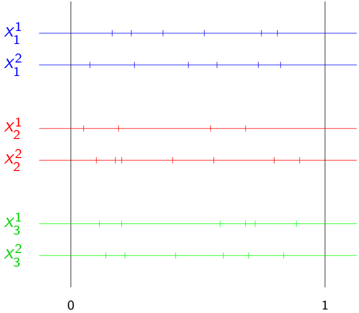
Theorem 2

Under similar assumptions as in theorem 1,

- *Asymptotic size:*
  Under (H_0),
  $$\mathbb{P}\left(\Phi^+_{h,\alpha} = 1\right) \underset{n\to+\infty}{\longrightarrow} \alpha.$$
- *Asymptotic power:*
  For any alternative P such that $\mathbb{E}[h(X_i, X_{i'})] > 0$,
  $$\mathbb{P}\left(\Phi^+_{h,\alpha} = 1\right) \underset{n\to+\infty}{\longrightarrow} 1.$$

Permutation approach inspired by Hoeffding (1952)

Given $\mathbb{X}_n = (X_i)_{1 \le i \le n}$, where $X_i = \left(X_i^1, X_i^2\right)$ i.i.d. $\sim P$ in $\mathcal{X} \times \mathcal{X}$.



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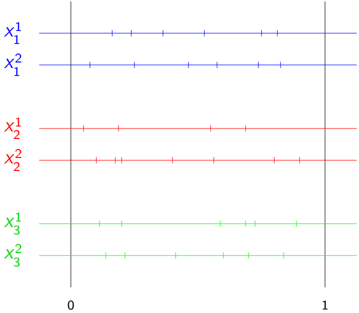
The permuted sample is, for $\Pi_n \sim \mathcal{U}(\mathfrak{S}_n)$ independent of $\mathbb{X}_n$

$$\mathbb{X}_n^{\Pi_n} = \left(X_1^{\Pi_n}, \dots, X_n^{\Pi_n}\right), \quad \text{with } X_i^{\Pi_n} = \left(X_i^1, X_{\Pi_n(i)}^2\right).$$



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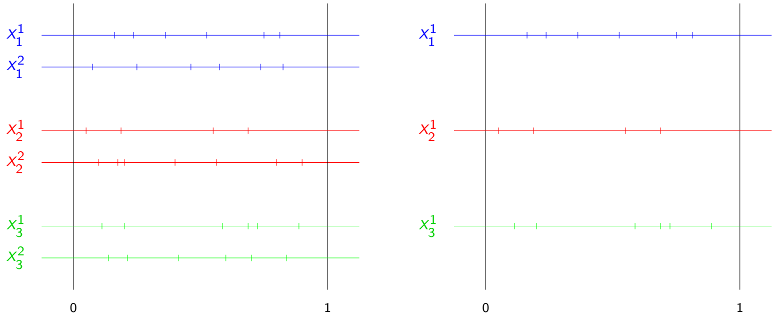

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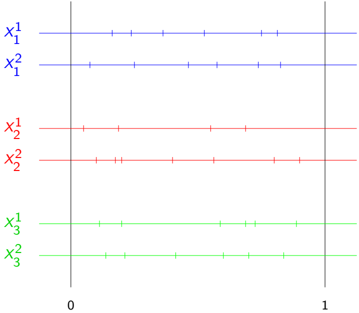

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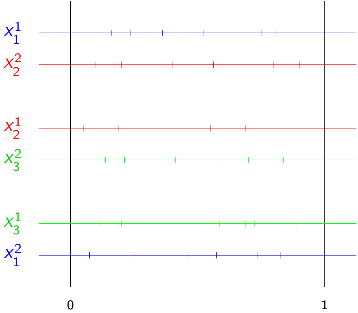

$\sqrt{n}U_{n,h}\left(\mathbb{X}_n^{\Pi_n}\right)$: on permuted data

Permutation approach inspired by Hoeffding (1952)

Given $\mathbb{X}_n = (X_i)_{1\leq i\leq n}$, where $X_i = \left(X_i^1, X_i^2\right)$ i.i.d. $\sim P$ in $\mathcal{X}\times\mathcal{X}$.

The permuted sample is, for $\Pi_n \sim \mathcal{U}\left(\mathfrak{S}_n\right)$ independent of $\mathbb{X}_n$

$$\mathbb{X}_n^{\Pi_n} = \left(X_1^{\Pi_n}, \ldots, X_n^{\Pi_n}\right), \quad \text{with } X_i^{\Pi_n} = \left(X_i^1, X_{\Pi_n(i)}^2\right).$$

Proposition

Let $\Pi_n \sim \mathcal{U}(\mathfrak{S}_n) \perp\!\!\!\perp \mathbb{X}_n$.

Under (H_0), $\mathbb{X}_n^{\Pi_n}$, and $\mathbb{X}_n$ have the same distribution.

Theorem 1

Let $h = h_\varphi$ with $\varphi : \mathcal{X}^2 \to \mathbb{R}$ satisfying $(\mathcal{A}_\varphi^{Mmt})$.

$$(\mathcal{A}_\varphi^{Mmt}) \left\{ \begin{array}{l} \text{for any } X \text{ with distribution } P \text{ or } P^1 \otimes P^2 \text{ on } \mathcal{X}^2, \\ \mathbb{E}\left[\varphi^4(X^1, X^2)\right] < +\infty. \end{array} \right.$$

Permutation approach

Consistency of the method

Theorem 1

Let $h = h_\varphi$ with $\varphi : \mathcal{X}^2 \to \mathbb{R}$ satisfying $(\mathcal{A}_\varphi^{Mmt})$.
Then, if $U_{n,h}$ is non-degenerate,

$$d_{W_2}\left(\mathcal{L}\left(\sqrt{n}U_{n,h_\varphi}\left(\mathbb{X}_n^{\Pi_n}\right)|\mathbb{X}_n\right), \mathcal{N}\left(0, \sigma_{\perp\!\!\!\perp}^2\right)\right) \xrightarrow[n\to+\infty]{\mathbb{P}} 0.$$

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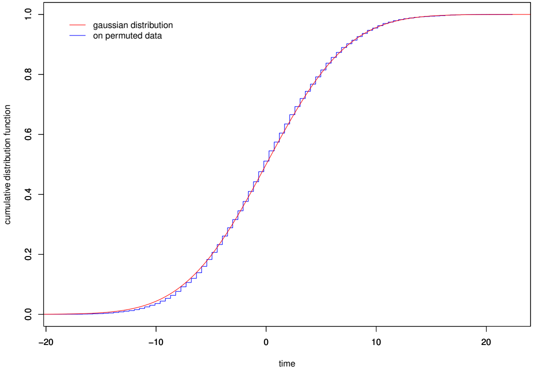

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Main arguments:

- Martingale difference array CLT ($\Rightarrow$ weak convergence).
- Second order moments convergence.

Permutation approach

Permutation independence test

Permutation test of independence

$$\Psi^{+}_{h,\alpha} = \mathbb{1}_{\left\{\sqrt{n}U_{n,h}(\mathbb{X}_n) > q^{\pi}_{h,1-\alpha}(\mathbb{X}_n)\right\}},$$

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Theorem 2

Under similar assumptions as in Theorem 1,

- *Non-asymptotic level / Asymptotic size:*
  Under (H_0),

$$\mathbb{P}\left(\Psi^{+}_{h,\alpha} = 1\right) \leq \alpha.$$

$$\mathbb{P}\left(\Psi^{+}_{h_\varphi,\alpha} = 1\right) \underset{n\to+\infty}{\longrightarrow} \alpha.$$

- *Asymptotic power:*
  Under any alternative P such that $\mathbb{E}[h_\varphi(X_i, X_{i'})] > 0$,

$$\mathbb{P}\left(\Psi^{+}_{h_\varphi,\alpha} = 1\right) \underset{n\to+\infty}{\longrightarrow} 1.$$

Monte Carlo approximation

In the previous tests, replacing the exact quantiles $q^*_{h,1-\alpha}(\mathbb{X}_n)$ and $q^\pi_{h,1-\alpha}(\mathbb{X}_n)$ by the corresponding empirical quantiles obtained from $B(n)$ i.i.d. bootstrapped/permuted samples from $\mathbb{X}_n$, with $B(n) \underset{n\to+\infty}{\longrightarrow} +\infty$ leads to the same theoretical results.

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Study on simulated and real data

Implementation (R and C++) $\Rightarrow$ fast computation.

Simulation study

Parameters

Simulated data

- Study of the level:
  - homogeneous Poisson processes on $[0.1, 0.2]$ with intensity $\lambda = 60$,
  - inhomogeneous Poisson processes with intensity $f_\lambda(t) = t \mapsto \lambda t$, $\lambda = 60$.
- Study of the power: injection model

$$X^j = X^j_{ind} \cup X_{com}, \quad \text{for } j = 1, 2$$

with $X^1_{ind} \perp\!\!\!\perp X^2_{ind}$ as above with parameter $\lambda_{ind} = 54$, and X_{com} a common homogeneous Poisson process with intensity $\lambda_{com} = 6$.

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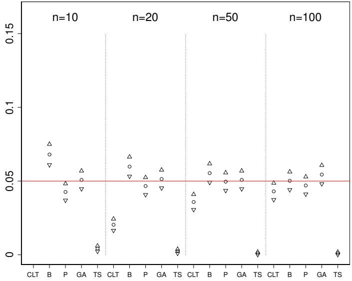
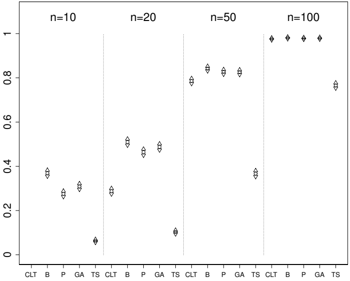
- n varies in $\{10, 20, 50, 100\}$,
- $\delta = 0.01$,
- $B = 10000$ step in the Monte Carlo approximation of the quantiles,
- $Ntest = 5000$ for the estimations of the size, and the power.

Simulation study

Results





Simulation study

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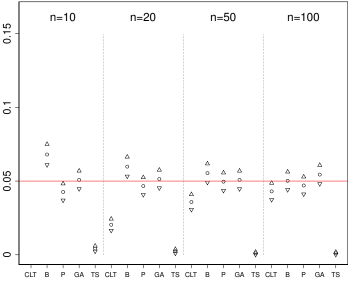

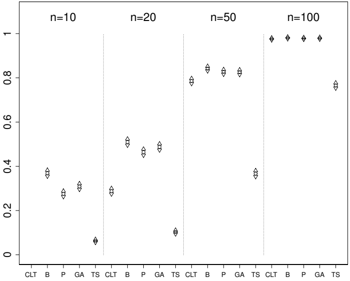

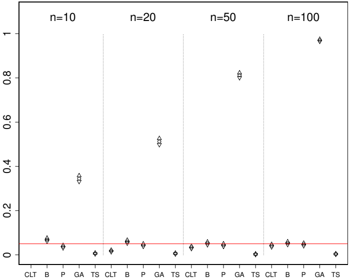

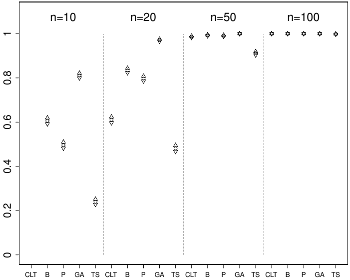

Multiple testing

Initial motivation

Detect the synchronizations.

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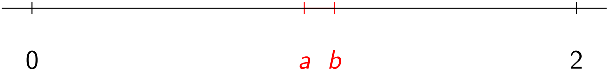
Idea: simultaneously test independence on sliding time windows $[a_k, b_k]$,



$$(\mathcal{H}_{0,k}) : X^1 \perp\!\!\!\perp X^2 \text{ on } [a_k, b_k] \quad \text{against } (\mathcal{H}_{1,k}) : X^1 \not\!\perp\!\!\!\perp X^2 \text{ on } [a_k, b_k].$$

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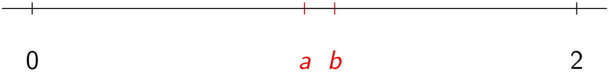

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Control the m tests at a global level α.

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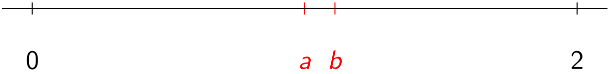

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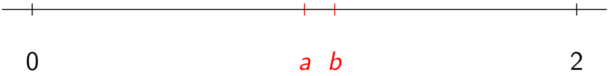

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$\hookrightarrow$ Benjamini and Hochberg multiple testing procedure to control the False Discovery Rate (1995).

Multiple testing procedure

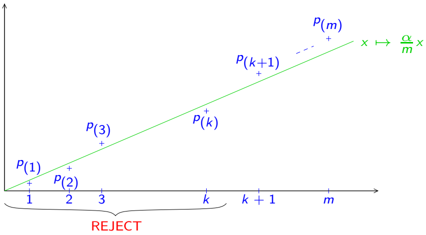
Benjamini-Hochberg procedure

- Sort the p-values

$$p_{(1)} \leq \cdots \leq p_{(m)},$$

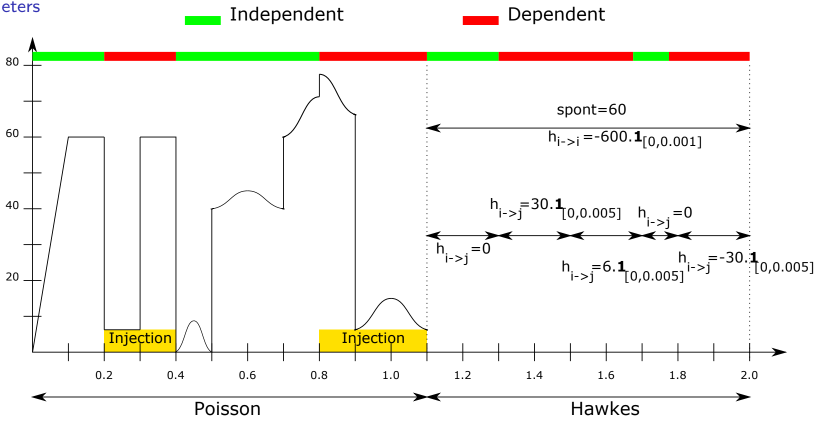
  where p_i corresponds to the test $(\mathcal{H}_{0,i})$,
- Reject all null hypotheses $\left\{\mathcal{H}_{(i)}\right\}_{1\leq i\leq k}$ where

$$k = \max\left\{i \ ; \ p_{(i)} \leq \frac{i}{m}q^*\right\}.$$



Simulation study

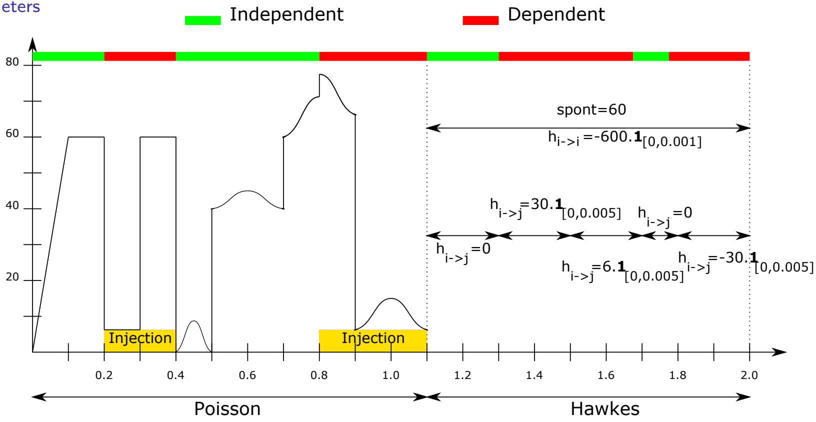
Parameters



A: Description of Experiment 1

Simulation study

Parameters

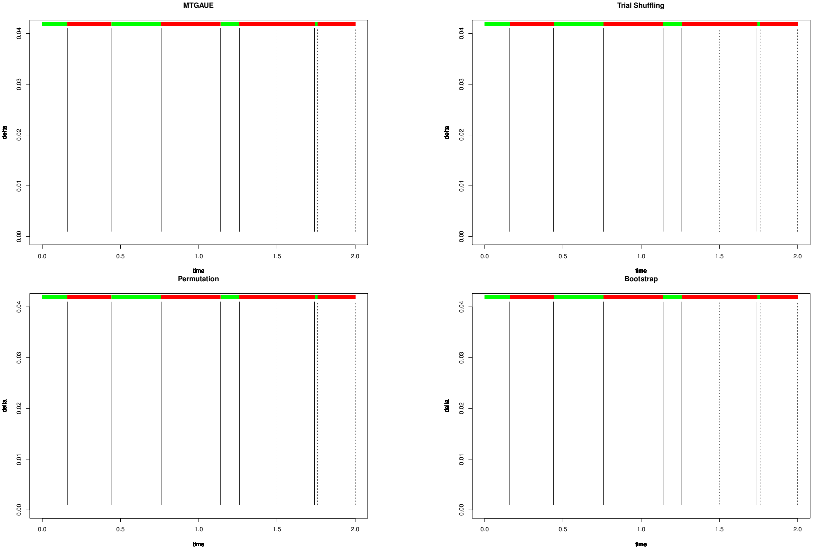


A: Description of Experiment 1

- $n = 50$,
- δ varies in $\{0.001, 0.002, \ldots, 0.04\}$,
- $B = 10000$ steps in the Monte Carlo approximation of the quantiles,
- $m = 191 \times 2$ simultaneous tests.

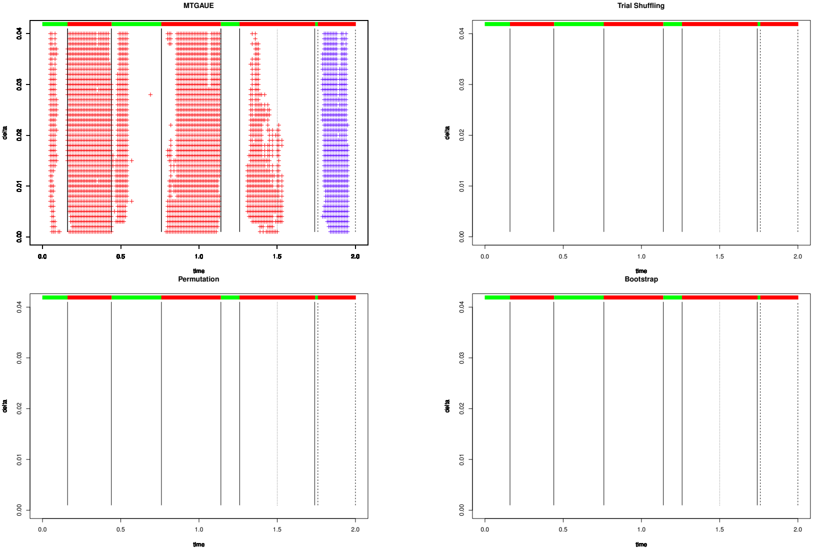
Simulation study

Results



Simulation study

Results



Simulation study

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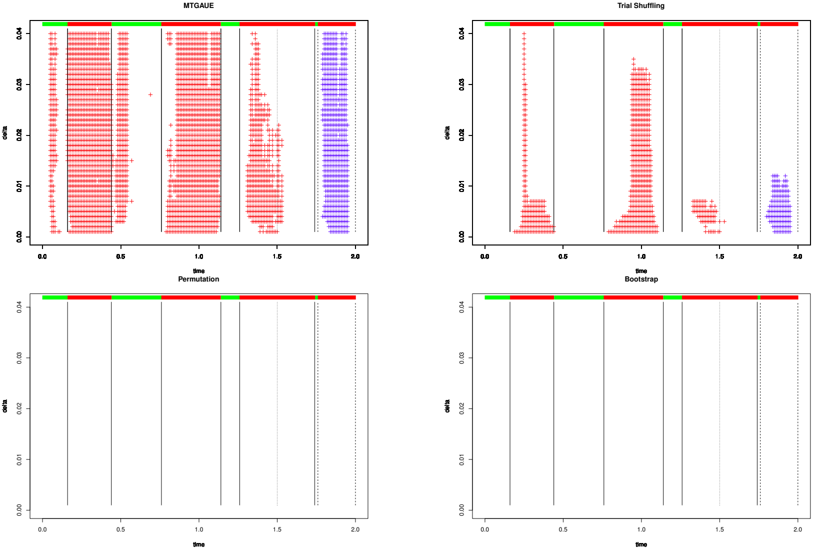

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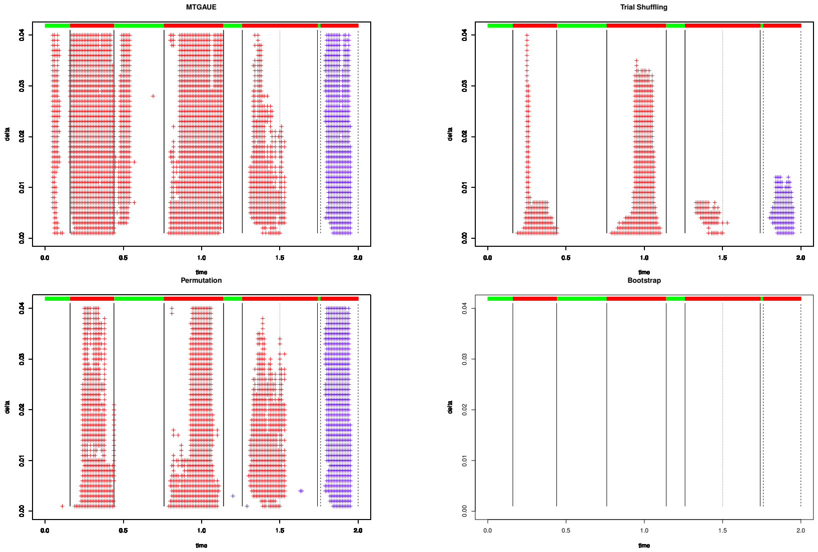

Simulation study

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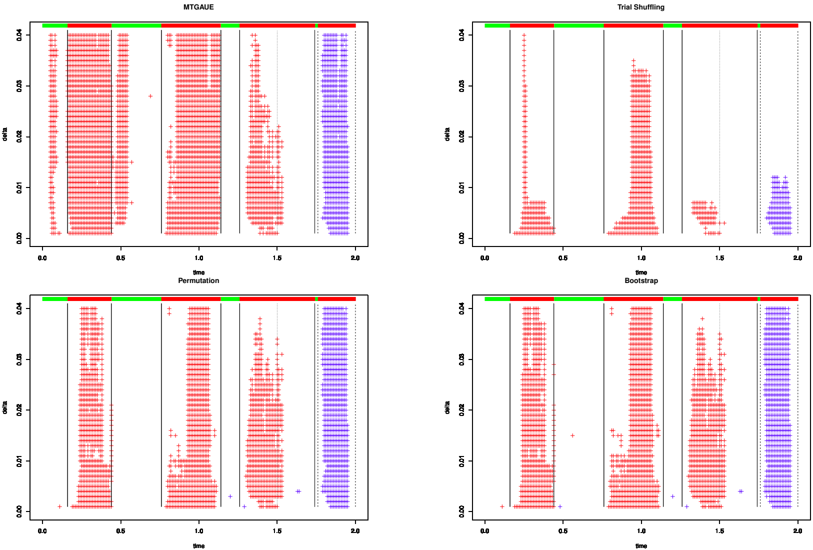
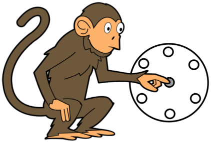


Real Data

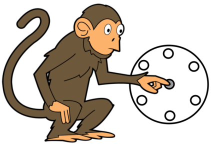
Experiment



Start

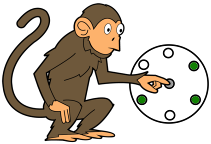
Real Data

Experiment



Start

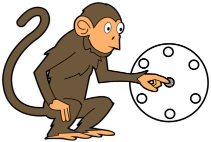
→ 500 ms



Preparatory signal

Real Data

Experiment



Start

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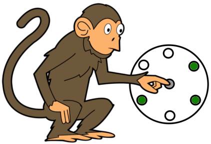
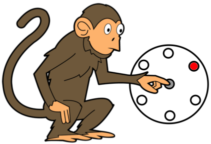


Preparatory signal

→ delay

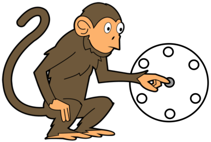


Response signal

with random delay $= \begin{cases} 600 \text{ ms} & \text{with probability } 0.3 \\ 1200 \text{ ms} & \text{otherwise} \end{cases}$

Real Data

Experiment



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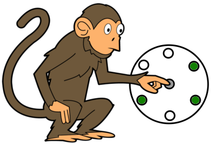

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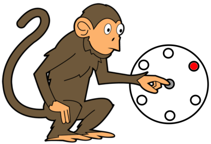


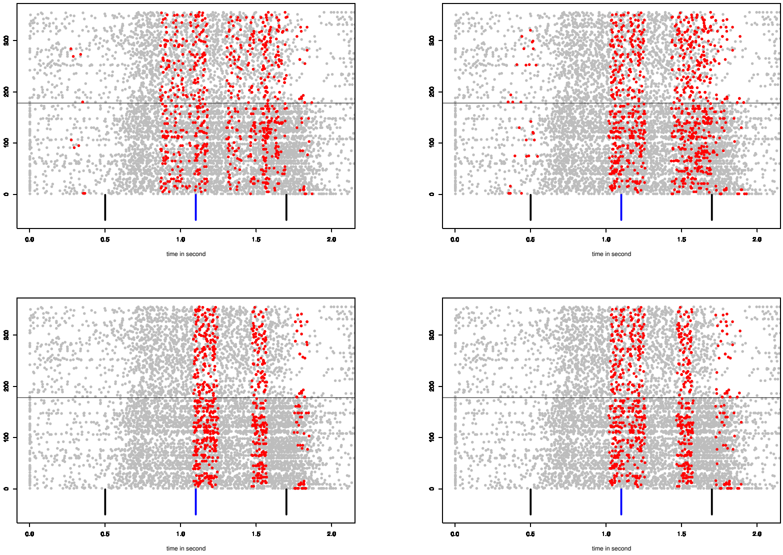
Start | Preparatory signal | Response signal

with random delay $= \begin{cases} 600 \text{ ms} & \text{with probability } 0.3 \\ 1200 \text{ ms} & \text{otherwise} \end{cases}$

Keep only trials with a response signal at 1700 ms.

Real Data

Results



Work still in progress and perspectives

Conclusions and perspectives

- Asymptotic performances of the bootstrap approach.
- Asymptotic/non-asymptotic performances of the permutation approach.

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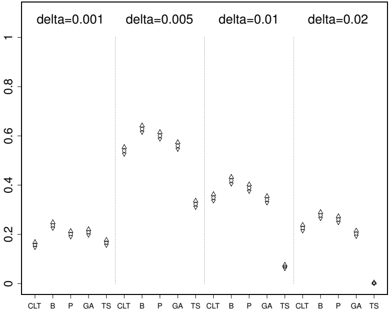

Work still in progress and perspectives

Conclusions and perspectives

- Asymptotic performances of the bootstrap approach.
- Asymptotic/non-asymptotic performances of the permutation approach.
- Choice of δ for the notion of coincidences?
- Theoretical justification of the multiple testing?

Merci !

Test statistic based on the total number of coincidences with delay:

$$C^{obs} = C(\mathbb{X}_n) = \sum_{i=1}^{n} \varphi_\delta^{coinc}\left(X_i^1, X_i^2\right).$$

General idea

Reject independence when there are too many (resp. too few) coincidences compared to what is expected under independence.

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Reject independence when there are too many (resp. too few) coincidences compared to what is expected under independence.

How to recreate the distribution under independence?

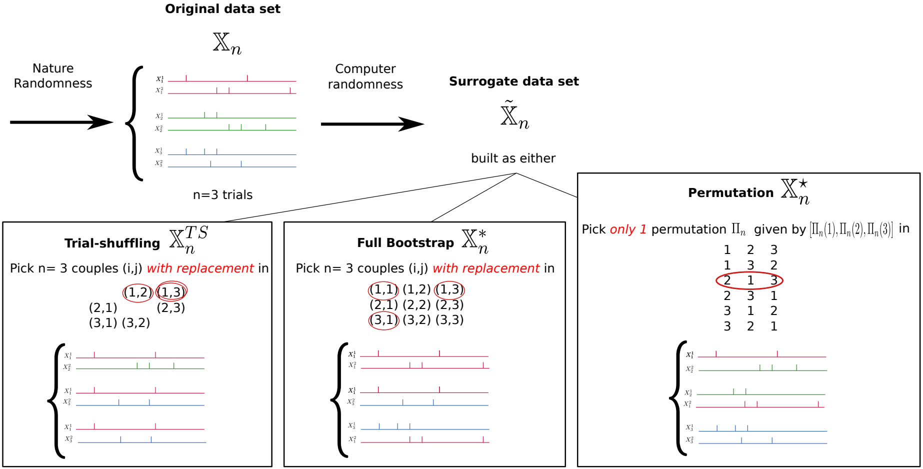
Construct a new sample $\tilde{\mathbb{X}}_n$ from the original one, i.e. $\mathbb{X}_n$, such that

$$\mathcal{L}\left(C\left(\tilde{\mathbb{X}}_n\right)\middle|\mathbb{X}_n\right) \quad \approx \quad \mathcal{L}\left(C\left(\mathbb{X}_n^{\perp\!\!\!\perp}\right)\right),$$

whether $\mathbb{X}_n$ satisfies independence or not.

The different resampling approaches

Trial Shuffling, Full Bootstrap and Permutation



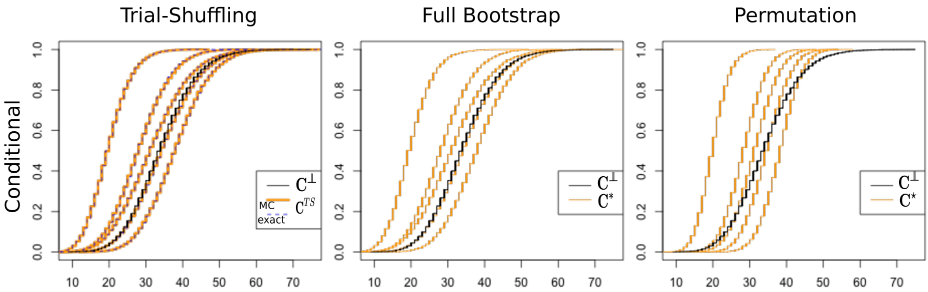
Unconditional distribution: *all* possible *choices* of both *Nature and Computer* randomness

Conditional distribution: *1 fixed* original data set (*Nature* randomness), *all* possible *choices* of *Computer* randomness

Conditional distributions of the number of coincidences

How do they perform?

$$\mathcal{L}\left(C\left(\tilde{\mathbb{X}}_n\right)\middle|\mathbb{X}_n\right) \quad \approx \quad \mathcal{L}\left(C\left(\mathbb{X}_n^{\perp\!\!\!\perp}\right)\right)?$$



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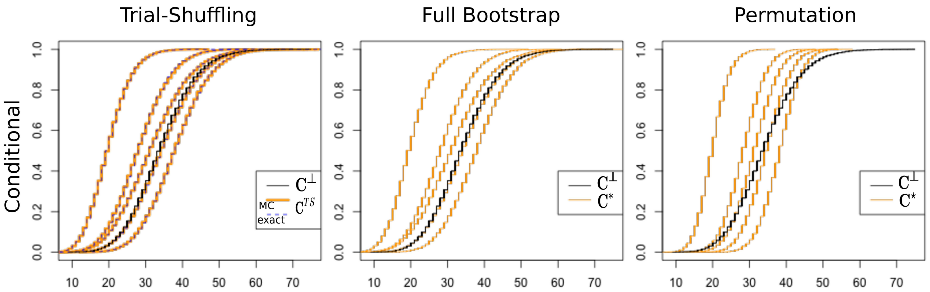

Centering issue !!!

Centering trick

In view of the statistical literature, it is not possible to estimate $\mathcal{L}\left(C\left(\mathbb{X}_n^{\perp\!\!\perp}\right)\right)$ directly, BUT,

$$\mathcal{L}\Big(C\big(\tilde{\mathbb{X}}_n\big) - \mathbb{E}\big[C\big(\tilde{\mathbb{X}}_n\big)\,|\mathbb{X}_n\big]\Big|\mathbb{X}_n\Big) \quad \approx \quad \mathcal{L}\Big(C\big(\mathbb{X}_n^{\perp\!\!\perp}\big) - \mathbb{E}\big[C\big(\mathbb{X}_n^{\perp\!\!\perp}\big)\big]\Big).$$

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$$\mathcal{L}\left(U\left(\tilde{\mathbb{X}}_n\right) - \mathbb{E}\left[U\left(\tilde{\mathbb{X}}_n\right)|\mathbb{X}_n\right]\middle|\mathbb{X}_n\right) \quad \approx \quad \mathcal{L}\left(U\left(\mathbb{X}_n^{\perp\!\!\perp}\right)\right).$$

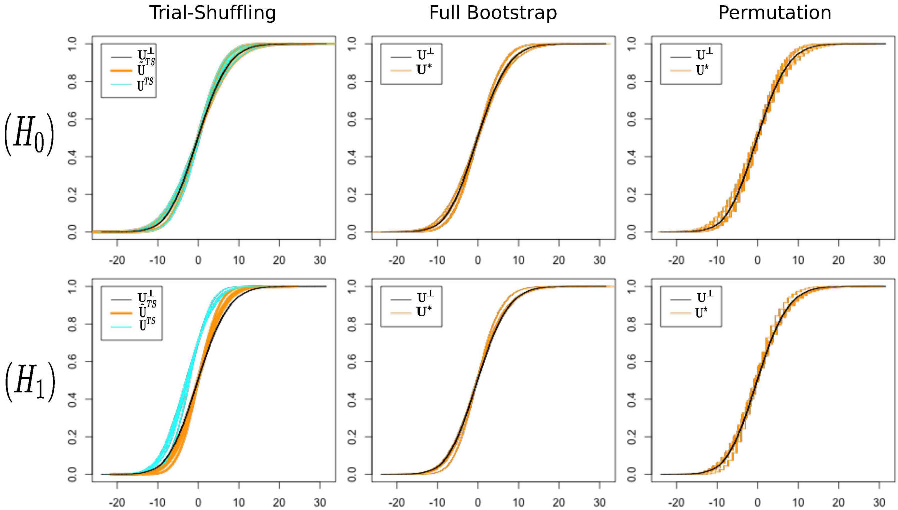
with

$$\mathbb{E}\left[U\left(\mathbb{X}_n^{TS}\right)\middle|\mathbb{X}_n\right] = -\frac{U(\mathbb{X}_n)}{n}, \text{ and } \begin{cases} \mathbb{E}[U(\mathbb{X}_n^*)|\mathbb{X}_n] = 0, \\ \mathbb{E}[U(\mathbb{X}_n^\star)|\mathbb{X}_n] = 0. \end{cases}$$

Conditional distributions of the centered number of coincidences

How do they perform?

$$\mathcal{L}\left(U\left(\tilde{\mathbb{X}}_n\right) - \mathbb{E}\left[U\left(\tilde{\mathbb{X}}_n\right) \middle| \mathbb{X}_n\right] \middle| \mathbb{X}_n\right) \quad \approx \quad \mathcal{L}\left(U\left(\mathbb{X}_n^{\perp\!\!\!\perp}\right)\right)?$$



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Critical value:

$(1-\alpha)$-quantile of $\mathcal{L}\left(U\left(\tilde{\mathbb{X}}_n\right) - \mathbb{E}\left[U\left(\tilde{\mathbb{X}}_n\right) | \mathbb{X}_n\right] \middle| \mathbb{X}_n\right)$,

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$\Rightarrow$ with Monte Carlo approximation.