Levenberg-Morrison-Marquardt (LMM) based on probabilistic models

El houcine Bergou

27/11/2015

Outline

Part 1

Problem statement

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The goal is to find the "best estimate" of the model states $X_0, X_1, \ldots$ knowing the observations $d_1, d_2, \ldots$ and the prior x_b.

Maximum a posteriori estimator

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Maximum a posteriori estimator is the minimizer of the following objective function

$$f(x_{0:k}) = \|F(x_{0:k})\|^2 = \| \underbrace{x_0 - x_b}_{\textbf{prior error}} \|^2_{\mathbf{B}^{-1}} + \sum_{i=1}^{k} \| \underbrace{x_i - \mathcal{M}_i(x_{i-1})}_{\textbf{model error}} \|^2_{\mathbf{Q}_i^{-1}}$$

$$+ \sum_{i=1}^{k} \| \underbrace{d_i - \mathcal{H}_i(x_i)}_{\textbf{observation error}} \|^2_{\mathbf{R}_i^{-1}}.$$

Gauss-Newton algorithm

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- Solve the linearized least-squares subproblem,

$$\min_{\delta x^j} \; m_j(x^j + \delta x^j) = \|F(x^j) + J_{F_j}\delta x^j\|^2,$$

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- Update $x^{j+1} = x^j + \delta x^j$.

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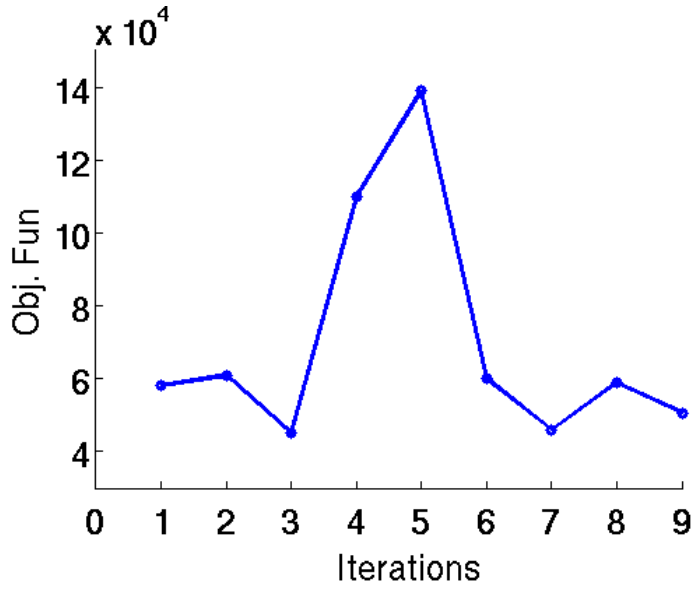

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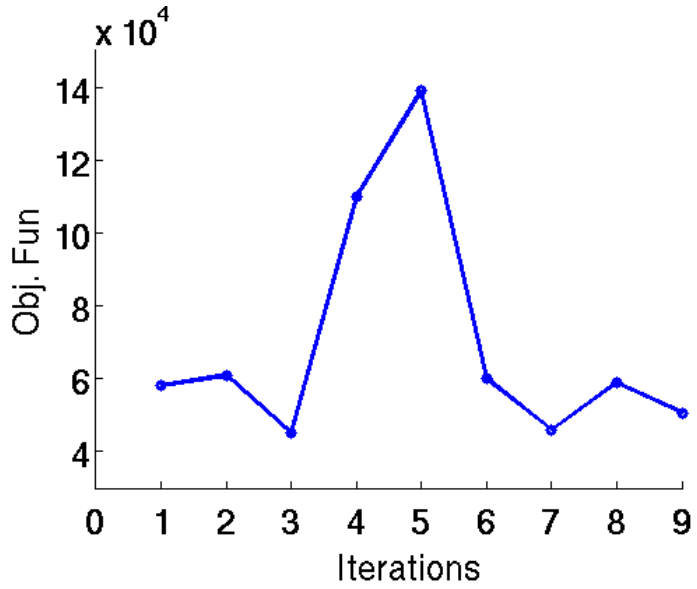

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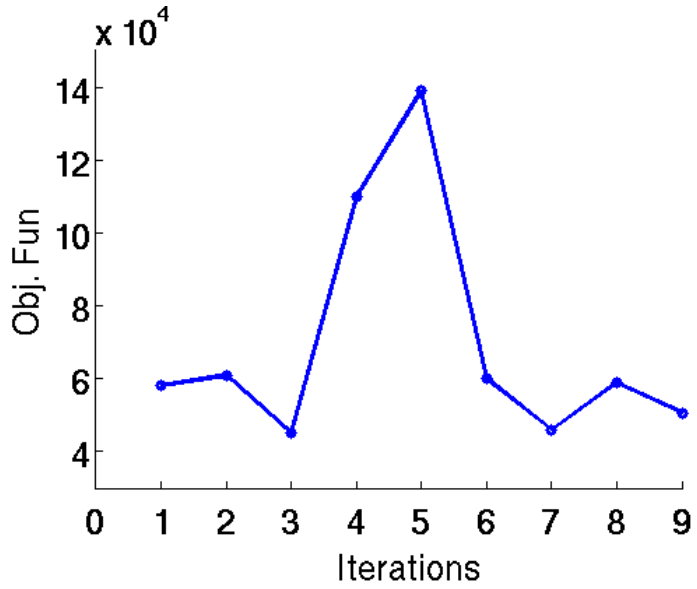

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- It is difficult to parallelize (speed-up, synchronization,...).

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2. Derivative-free.
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3. Handle the cases where the derivative approximations are random.

Part 2

LMM based on probabilistic models

Levenberg-Morrison-Marquardt method

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$\Rightarrow$ The Gauss-Newton subproblem

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$\Rightarrow$ The Gauss-Newton subproblem is replaced by

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- γ large $\Rightarrow \delta x^j \simeq -\frac{g_{f_j}}{\gamma^2}$ (small step).

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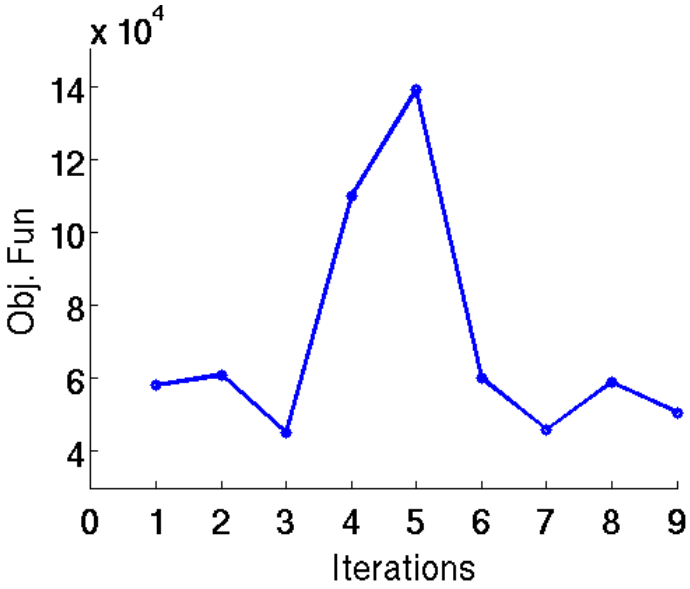
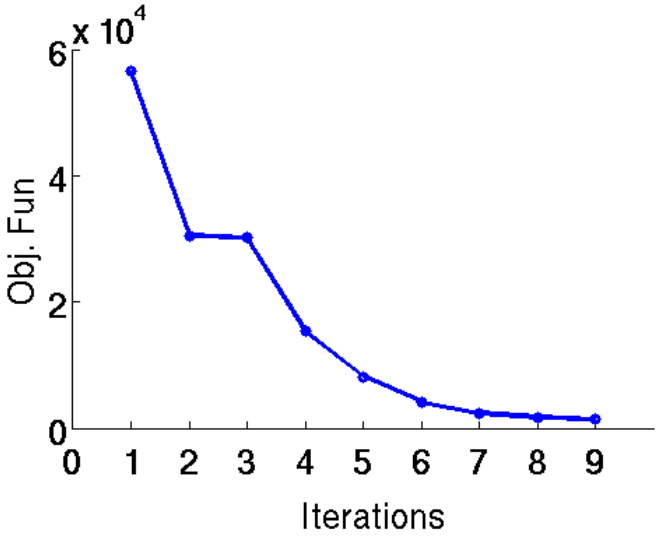
Numerical experiments



Gauss-Newton iterates



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$\Rightarrow$ what do we need from the model to ensure convergence ?

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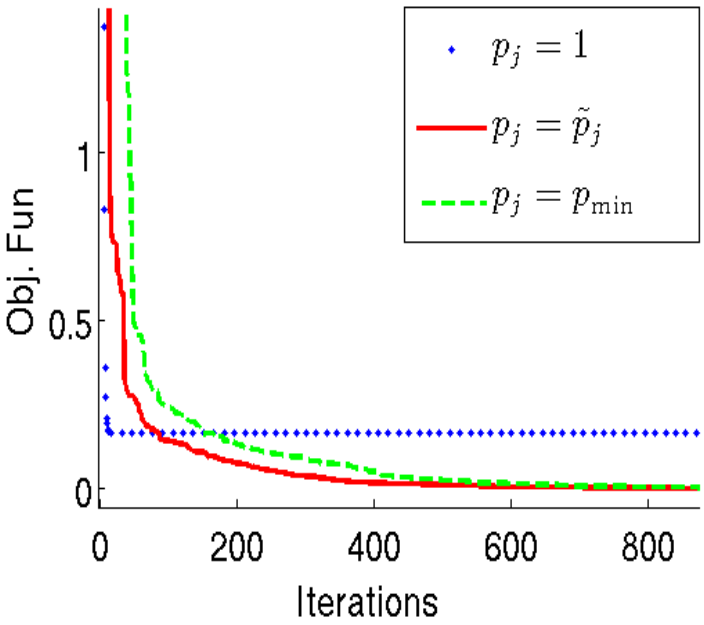
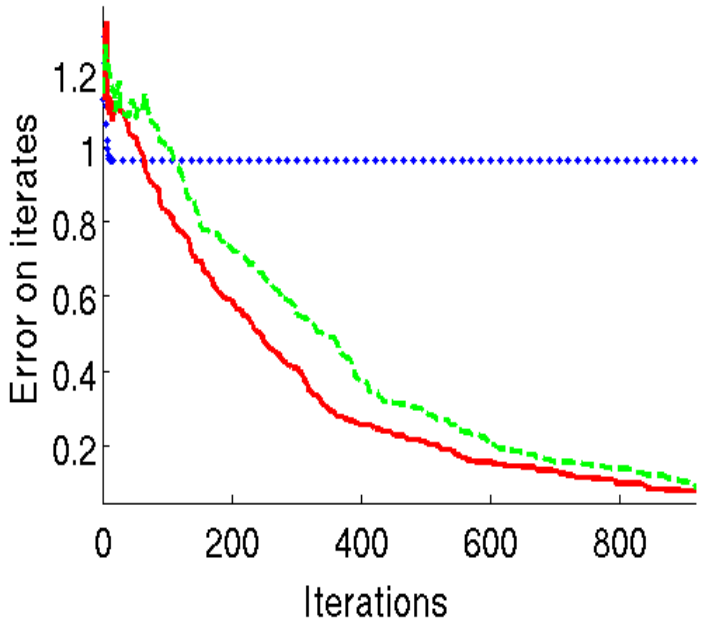


Average of function values



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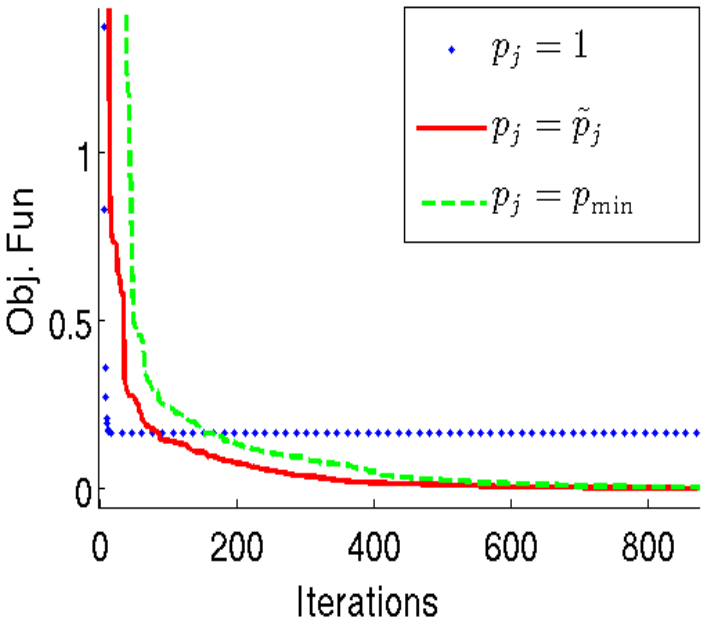

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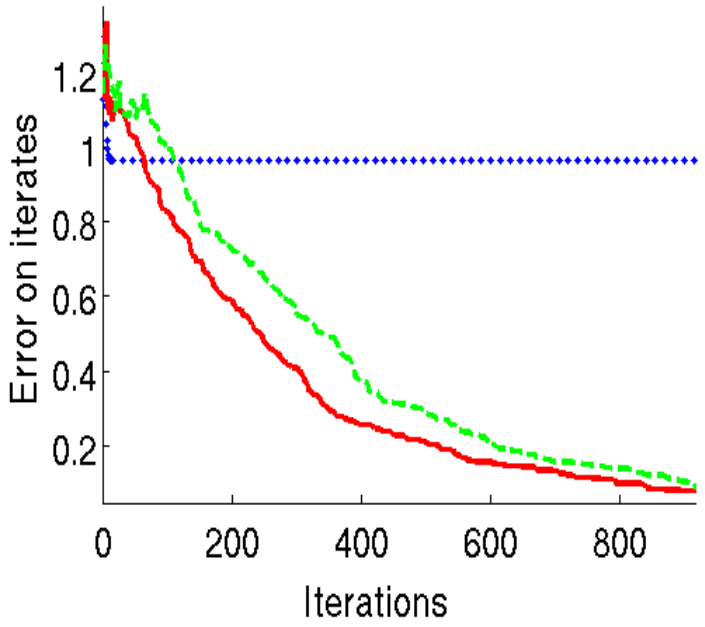

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Part 3

Conclusions

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1. We have derived a variant of the Levenberg-Morrison-Marquardt method to handle the scenarios where the linearized subproblem is solved inexactly and/or the gradient model is accurate only within a certain probability.
2. Under appropriate assumptions we have shown that our approach is globally convergent.
3. We have given some practical situations where our approach can be used.

Thank you for your attention.