# Improved Bounded Max Sum for Distributed Constraint Optimization

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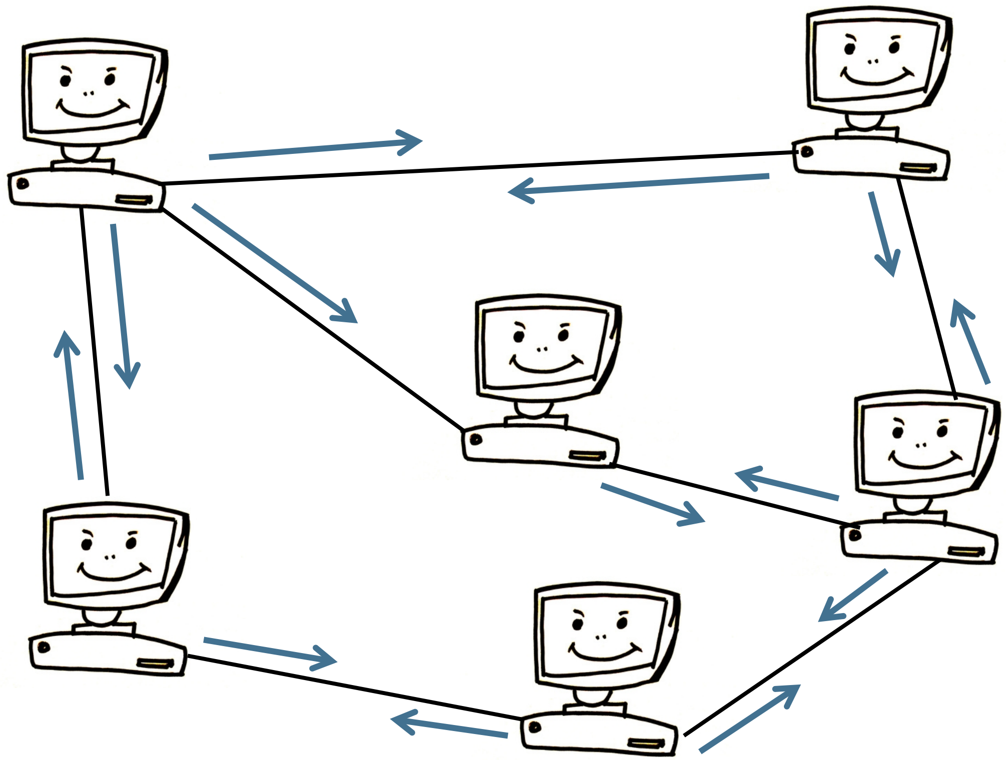
# Introduction



Context:
- Low-power devides
- Very fast response

We cannot find the optimal solution, but just a good solution

Approximation ratio (AR):
AR = UB / LB

Bad approximation ratio:
1. Bad solution (LB)
2. Bad UB

**Contribution**

**We propose an improved version of Bounded Max-Sum:**
1. **It improves the approximation ratio**
2. **It improves the approximate solution**

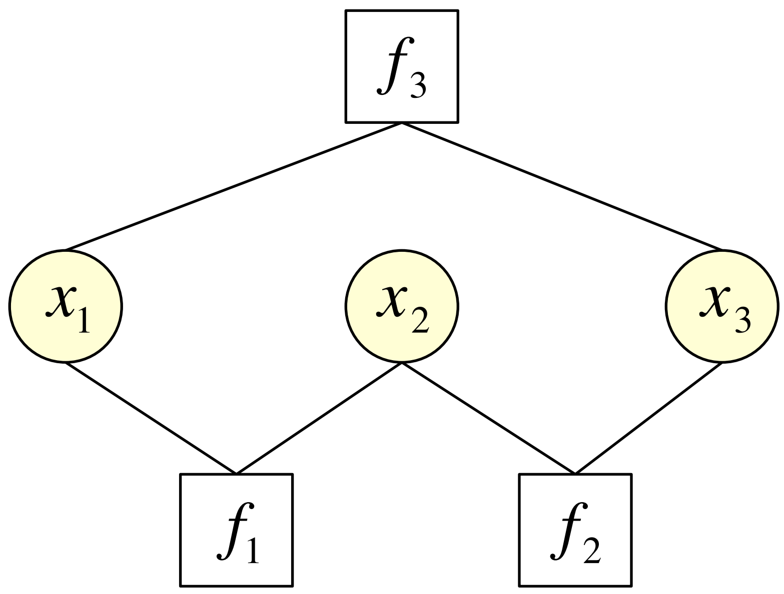
# DCOP

- $X = \{x_1, \ldots, x_n\}$: a set of variables
- $D = \{D_1, \ldots, D_n\}$: a set of domain values
- $F = \{f_1, \ldots, f_e\}$: a set of cost functions
- $A = \{a_1, \ldots, a_r\}$: a set of agents
- $\beta : F \rightarrow A$: a mapping between cost functions to agents

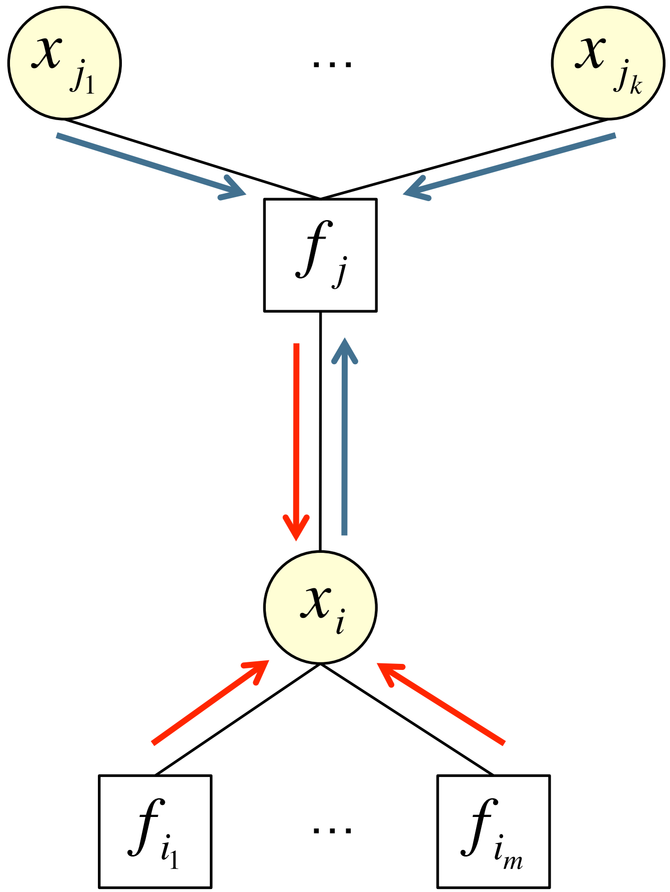
- Objective function: $F(X) = \sum_{f \in F} f(X)$
- Task: $x^* = \arg\max_X F(X)$
- Approximation ratio: $F(x) \leq F(x^*) \leq \rho F(x)$

# Factor Graph

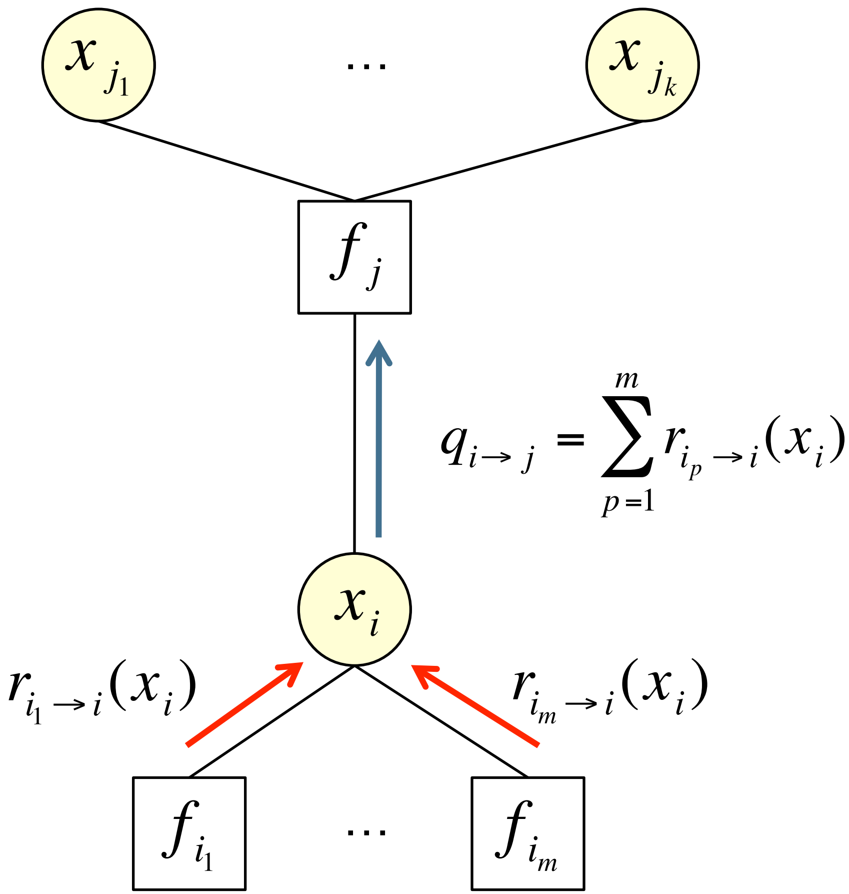
$$\{\, f_1(x_1,x_2),\ f_2(x_2,x_3),\ f_3(x_1,x_3)\, \}$$



# Max-Sum

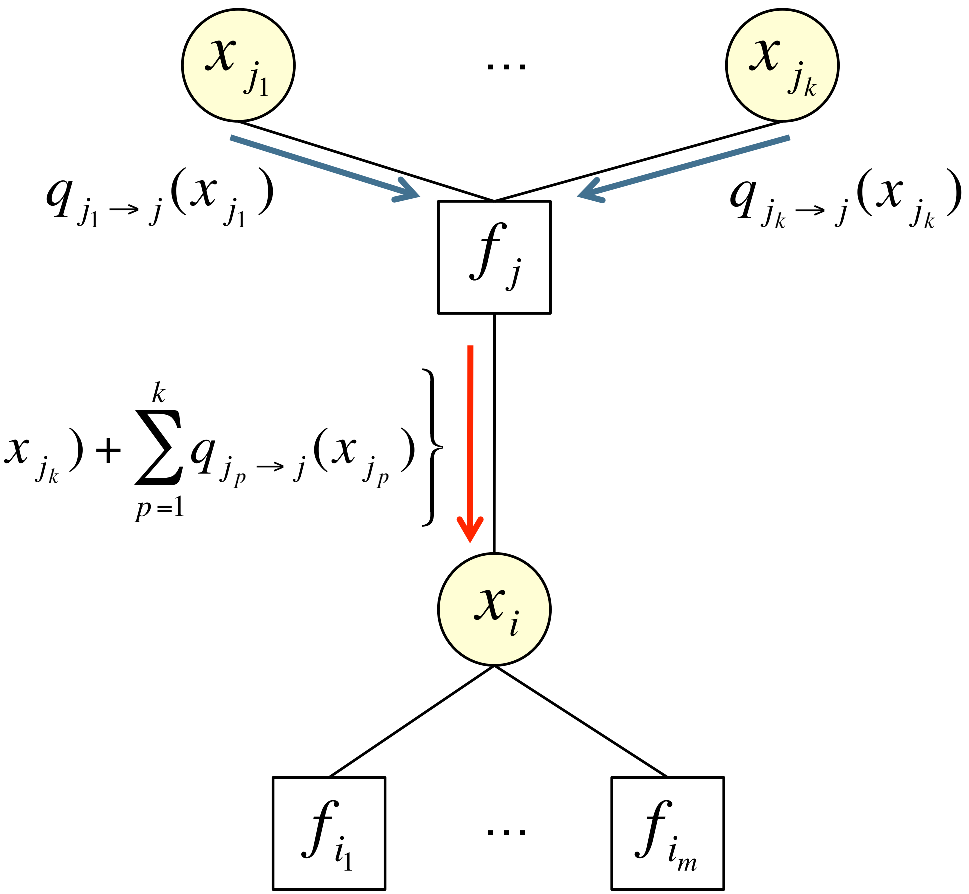


# Max-Sum



$x_{j_1}$
...
$x_{j_k}$
$f_j$
$q_{i \to j} = \sum_{p=1}^{m} r_{i_p \to i}(x_i)$
$x_i$
$r_{i_1 \to i}(x_i)$
$r_{i_m \to i}(x_i)$
$f_{i_1}$
...
$f_{i_m}$

# Max-Sum




$$r_{j \to i}(x_i) = \max_{x_{j_1}, \ldots, x_{j_k}} \left\{ f_j(x_i, x_{j_1}, \ldots, x_{j_k}) + \sum_{p=1}^{k} q_{j_p \to j}(x_{j_p}) \right\}$$

Max eliminates variables from the problem

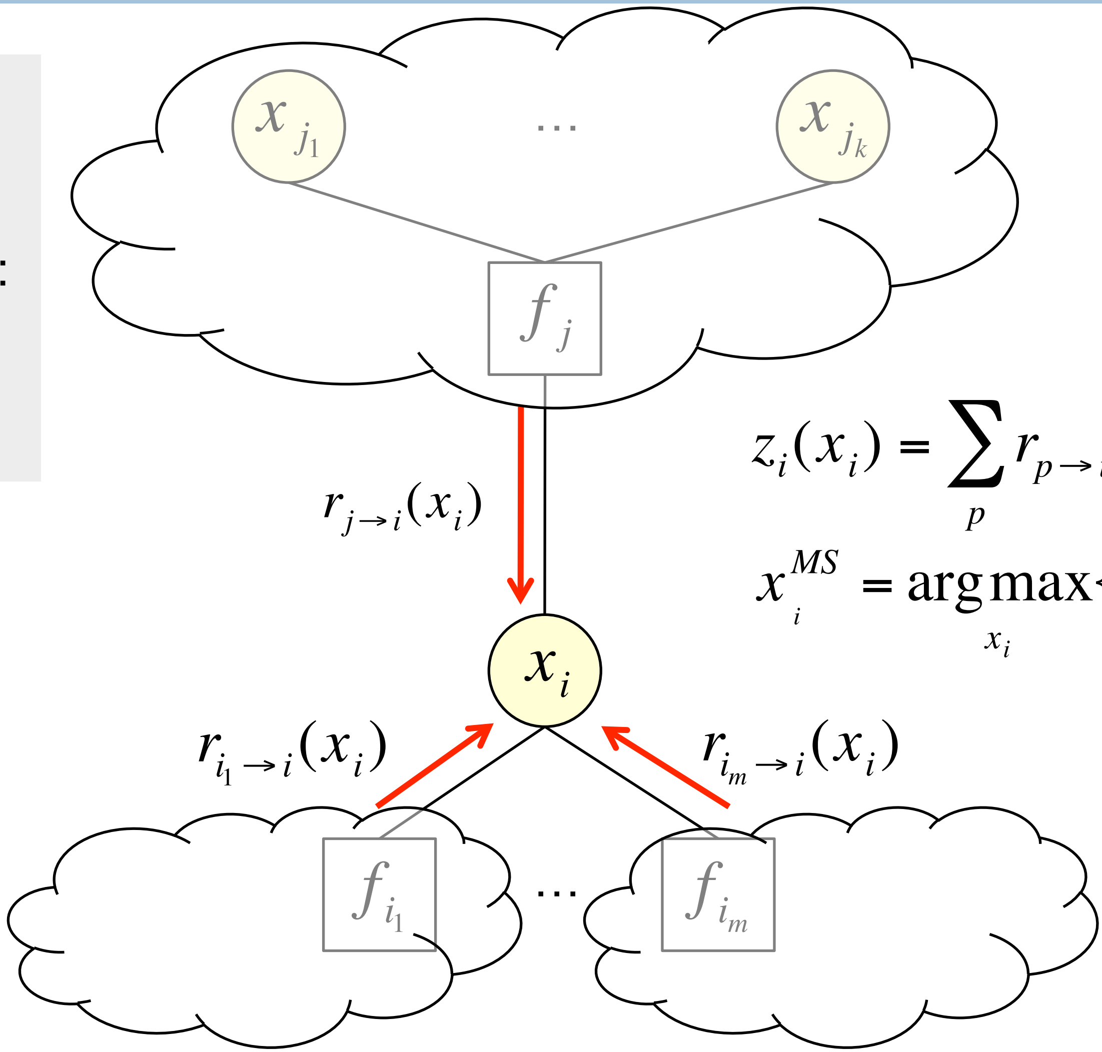
# Max-Sum

Always:

$$F(x^{MS}) \leq F(x^*)$$

If the factor graph is a tree:

$$F(x^{MS}) = F(x^*)$$

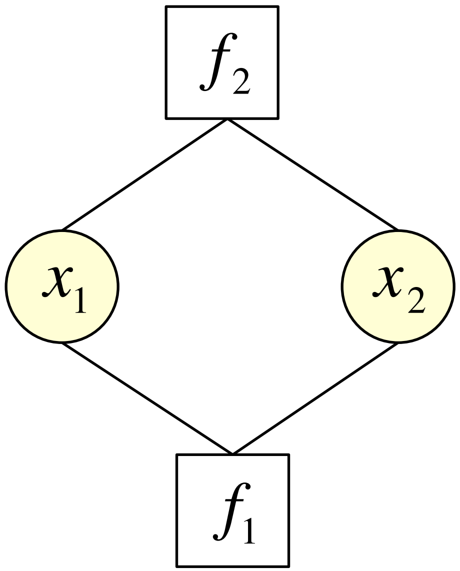



$$z_i(x_i) = \sum_p r_{p\to i}(x_i)$$

$$x_i^{MS} = \arg\max_{x_i} \left\{ z(x_i) \right\}$$

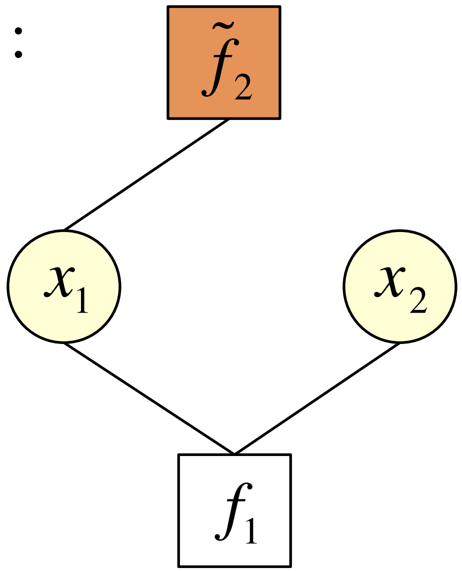
# Bounded Max-Sum (BMS)

$P$ :



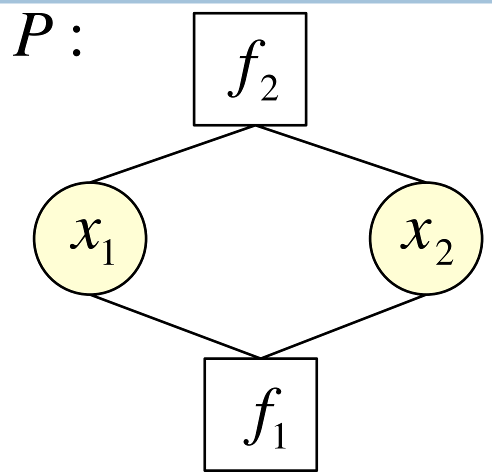

1. Remove cycles:

$\tilde{P}$ :



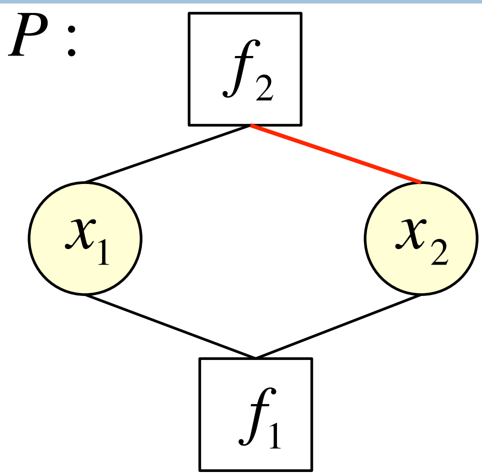

2. Solve $\tilde{P}$ using Max-Sum
3. Bound the optimum solution

# BMS: remove cycles

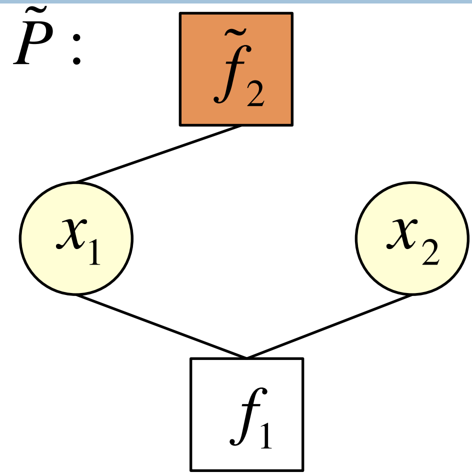



1. Select an edge to remove
2. Remove selected edge

# BMS: remove cycles




| $x_1$ | $x_2$ | $f_2(x_1, x_2)$ |
|---|---|---|
| a | a | 10 |
| a | b | 10 |
| b | a | 0 |
| b | b | 10 |




1. Select an edge to remove

2. Remove selected edge

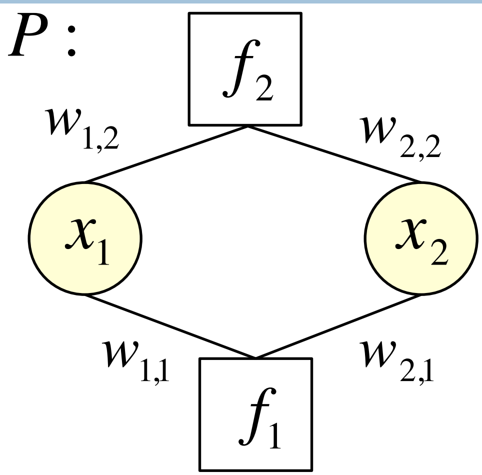
$\min_{x_2}$

| $x_1$ | $x_2$ | $f_2(x_1, x_2)$ |
|---|---|---|
| a | a | 10 |
| a | b | 10 |
| b | a | 0 |
| b | b | 10 |

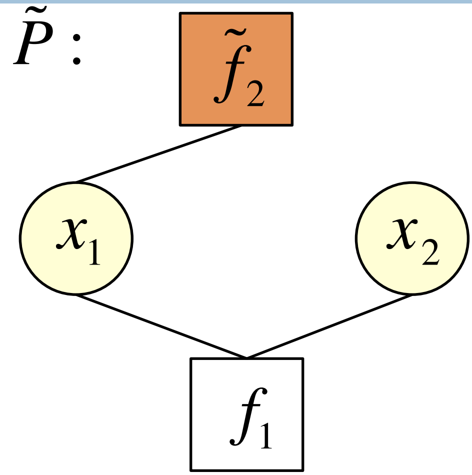
=

| $x_1$ | $\tilde{f}_2(x_1)$ |
|---|---|
| a | 10 |
| b | 0 |

# BMS: remove cycles




| $x_1$ | $x_2$ | $f_2(x_1, x_2)$ |
|---|---|---|
| a | a | 10 |
| a | b | 10 |
| b | a | 0 |
| b | b | 10 |




1. Select an edge to remove

2. Remove selected edge

Maximum cost lost by removing that edge

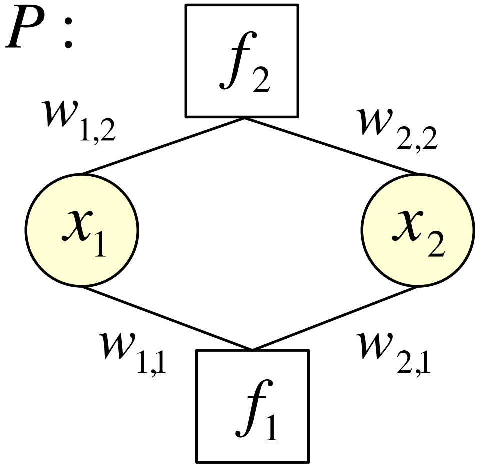
$$w_{2,2} = \max_{x_1} \left\{ \max_{x_2} \left[ f_2(x_1, x_2) \right] - \min_{x_2} \left[ f_2(x_1, x_2) \right] \right\}$$

| $x_1$ | $x_2$ | $f_2(x_1, x_2)$ |
|---|---|---|
| a | a | 10 |
| a | b | 10 |
| b | a | 0 |
| b | b | 10 |

| $x_1$ | $x_2$ | $f_2(x_1, x_2)$ |
|---|---|---|
| a | a | 10 |
| a | b | 10 |
| b | a | 0 |
| b | b | 10 |

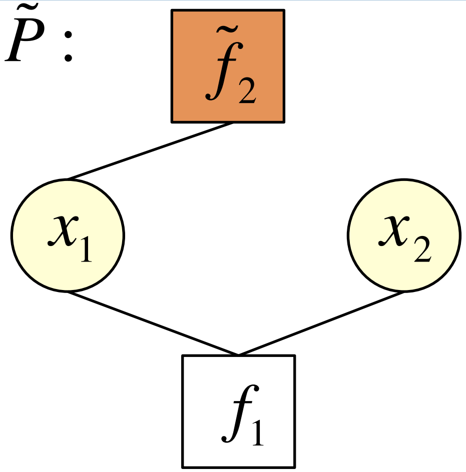
# BMS: remove cycles

$P:$




| $x_1$ | $x_2$ | $f_2(x_1, x_2)$ |
|---|---|---|
| a | a | 10 |
| a | b | 10 |
| b | a | 0 |
| b | b | 10 |

$\tilde{P}:$




## 1. Select an edge to remove

## 2. Remove selected edge

Maximum cost lost by removing that edge

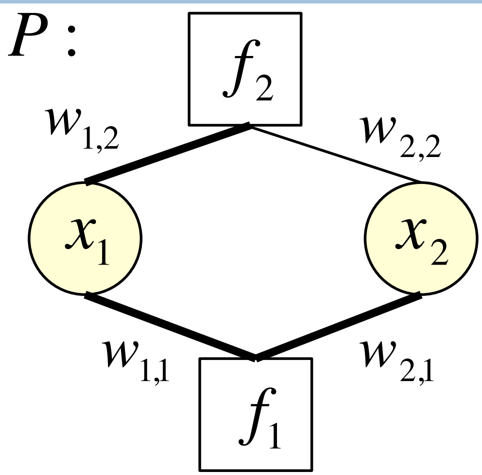
| $x_1$ | $x_2$ | $f_2(x_1, x_2)$ |
|---|---|---|
| a | a | 10 |
| a | b | 10 |
| b | a | 0 |
| b | b | 10 |

$\leq$

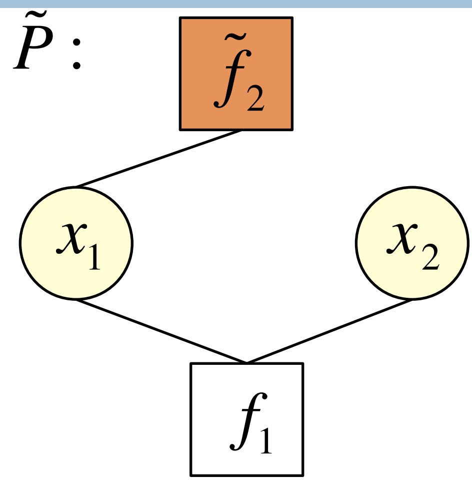
| $x_1$ | $\tilde{f}_2(x_1)$ |
|---|---|
| a | 10 |
| b | 0 |

$+ \; w_{2,2}$

# BMS: remove cycles




| $x_1$ | $x_2$ | $f_2(x_1, x_2)$ |
|---|---|---|
| a | a | 10 |
| a | b | 10 |
| b | a | 0 |
| b | b | 10 |




1. Select an edge to remove

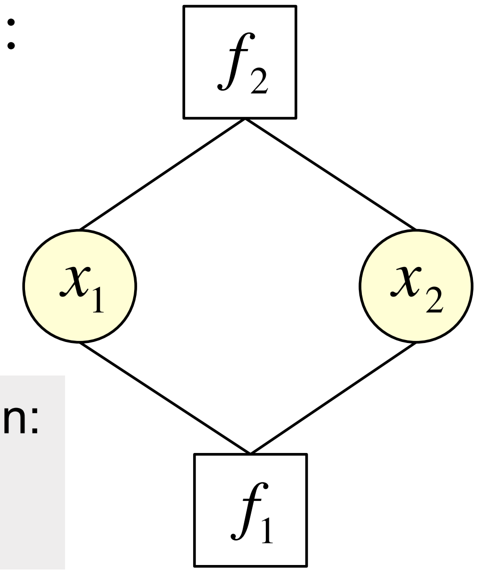
2. Remove selected edge

Maximum cost lost by removing that edge

- Compute a maximum spanning tree
- Eliminate edges not in the spanning tree

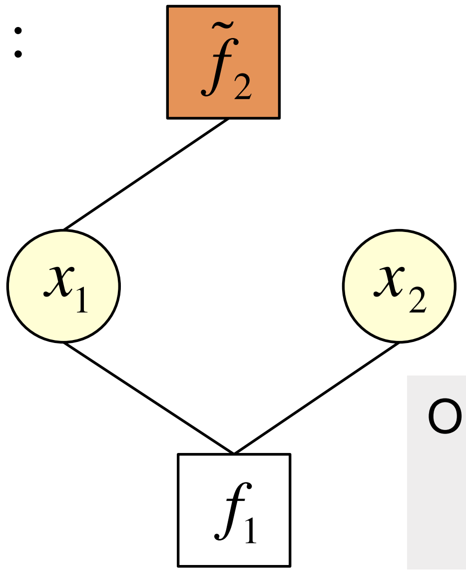
# BMS: bound the solution

$P :$




Optimal solution:
$x^*$

$\tilde{P} :$

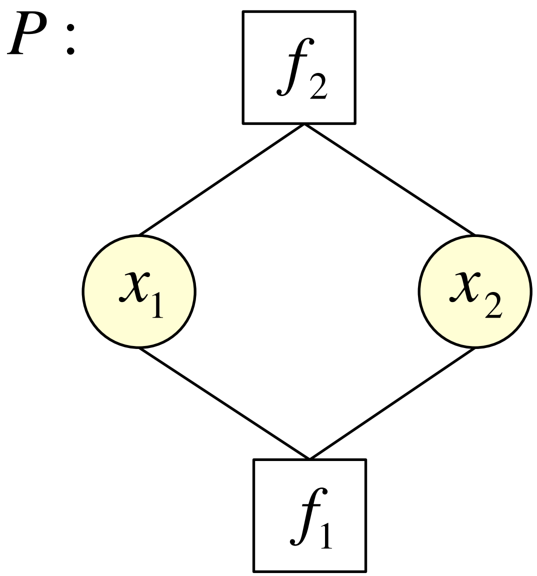



Optimal solution:
$x^{BMS}$

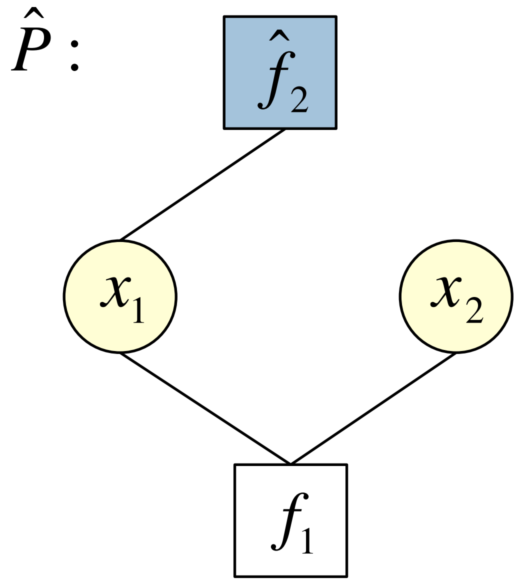
$$F(x^{BMS}) \le F(x^*) \le \tilde{F}(x^{BMS}) + W$$

$$F(x^{BMS}) \le F(x^*) \le \underbrace{\boxed{\frac{\tilde{F}(x^{BMS}) + W}{F(x^{BMS})}}}_{\tilde{\rho}} F(x^{BMS})$$

# Weak Improved Bounded Max-Sum (wIBMS)



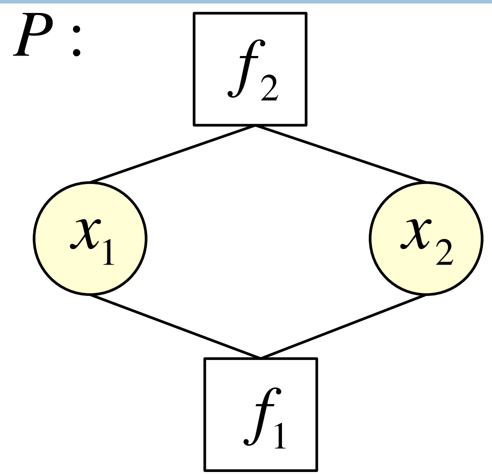

1. Remove cycles:




2. Solve using Max-Sum

3. Bound the optimum solution

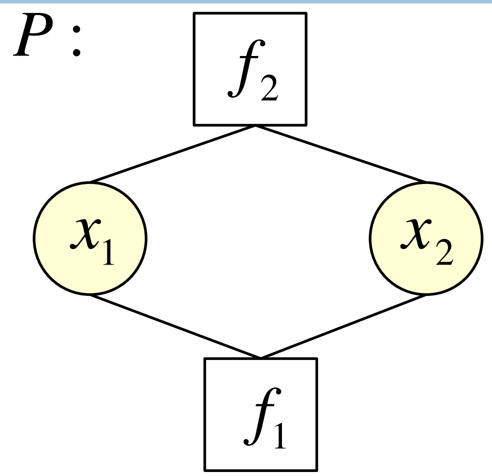
# wIBMS: remove cycles




1. Select an edge to remove

2. Remove selected edge

# wIBMS: remove cycles

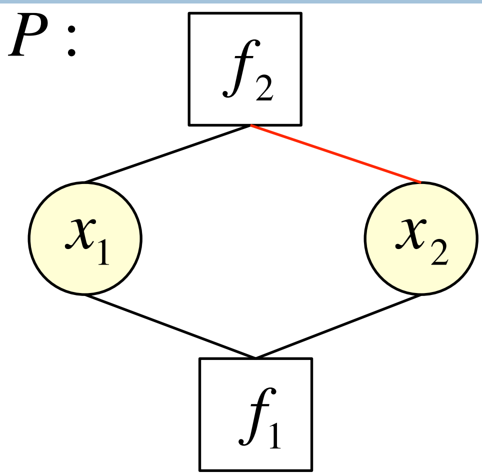



1. Select an edge to remove

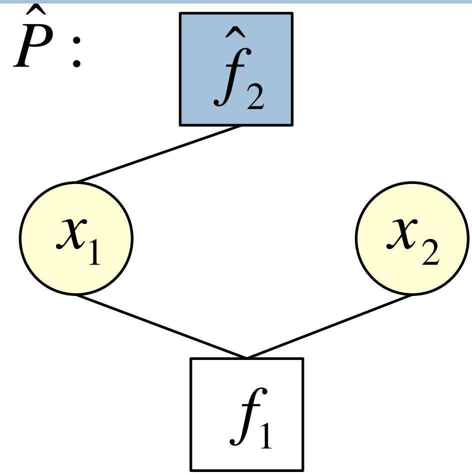
   Same procedure as BMS

2. Remove selected edge

# wIBMS: remove cycles




| $x_1$ | $x_2$ | $f_2(x_1, x_2)$ |
|---|---|---|
| a | a | 10 |
| a | b | 10 |
| b | a | 0 |
| b | b | 10 |



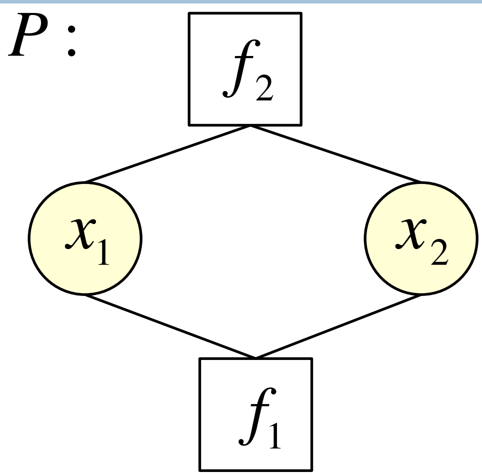

1. Select an edge to remove

2. Remove selected edge

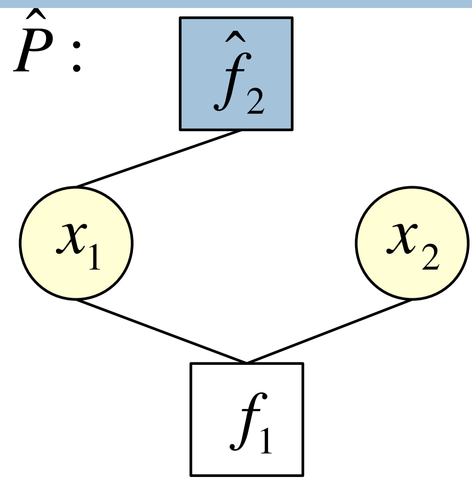
$$\max_{x_2} \left\{ \begin{array}{c|c|c} x_1 & x_2 & f_2(x_1, x_2) \\ \hline a & a & 10 \\ a & b & 10 \\ b & a & 0 \\ b & b & 10 \end{array} \right\} =$$

| $x_1$ | $\hat{f}_2(x_1)$ |
|---|---|
| a | 10 |
| b | 10 |

# wIBMS: remove cycles




| $x_1$ | $x_2$ | $f_2(x_1, x_2)$ |
|---|---|---|
| a | a | 10 |
| a | b | 10 |
| b | a | 0 |
| b | b | 10 |




1. Select an edge to remove

2. Remove selected edge

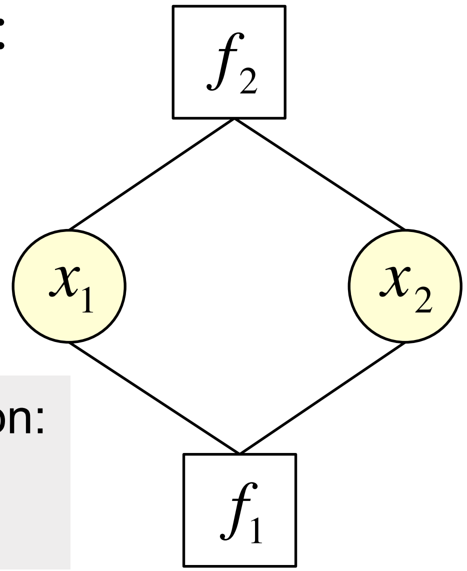
| $x_1$ | $x_2$ | $f_2(x_1, x_2)$ |
|---|---|---|
| a | a | 10 |
| a | b | 10 |
| b | a | 0 |
| b | b | 10 |

$\leq$

| $x_1$ | $\hat{f}_2(x_1)$ |
|---|---|
| a | 10 |
| b | 10 |

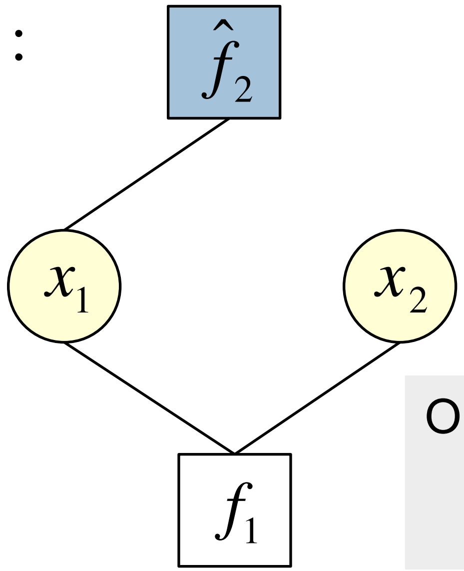
# wIBMS: bound the solution

$P:$




Optimal solution:
$x^*$

$\hat{P}:$




Optimal solution:
$x^w$

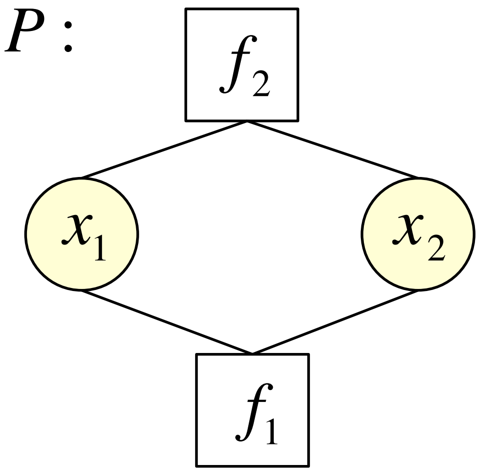
$$F(x^w) \le F(x^*) \le \hat{F}(x^w)$$

$$F(x^w) \le F(x^*) \le \underbrace{\boxed{\frac{\hat{F}(x^w)}{F(x^w)}}}_{\hat{\rho}} F(x^w)$$

# Relation BMS, wIBMS and IBMS

**Original**

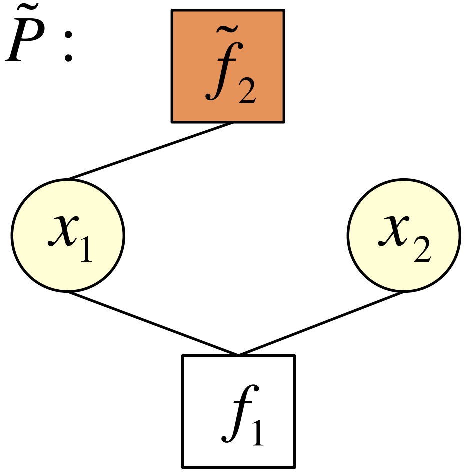
$P:$




$F(x^*)$

**BMS**

$\tilde{P}:$

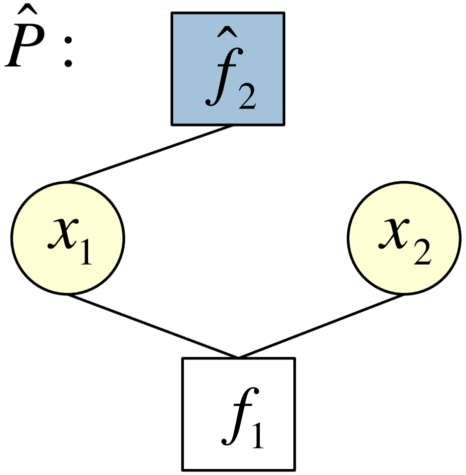



$LB : F(x^{BMS})$

$UB : \tilde{F}(x^{BMS}) + W$

**weak IBMS**

$\hat{P}:$




$LB : F(x^w)$

$UB : \hat{F}(x^w)$

$UB^{wIBMS} \leq UB^{BMS}$

$LB^{wIBMS} <> LB^{BMS}$

$\longrightarrow$

**IBMS**

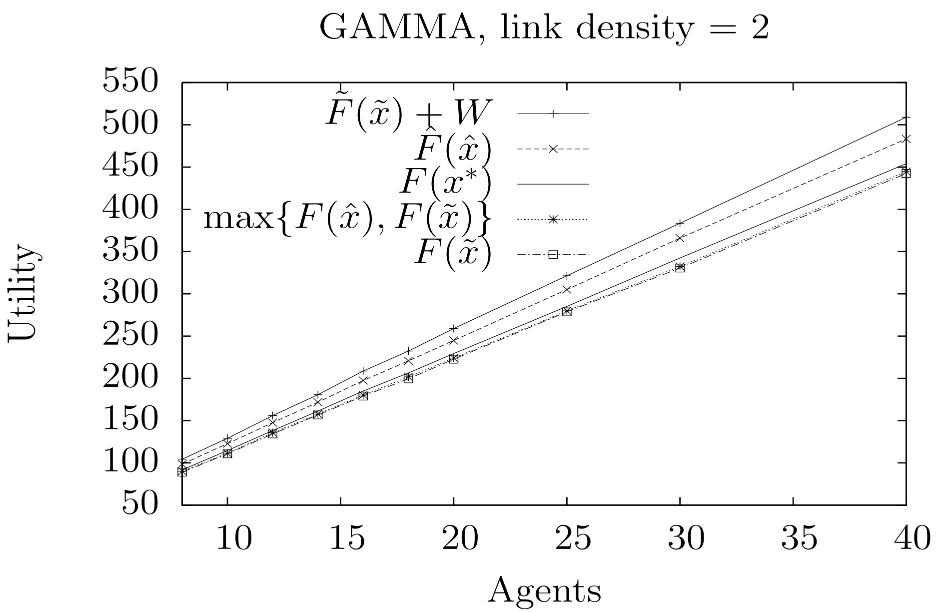
$$\rho = \frac{UB^{wIBMS}}{\max\{LB^{BMS}, LB^{wIBMS}\}} \begin{matrix} \leq \tilde{\rho} \\ \leq \hat{\rho} \end{matrix}$$

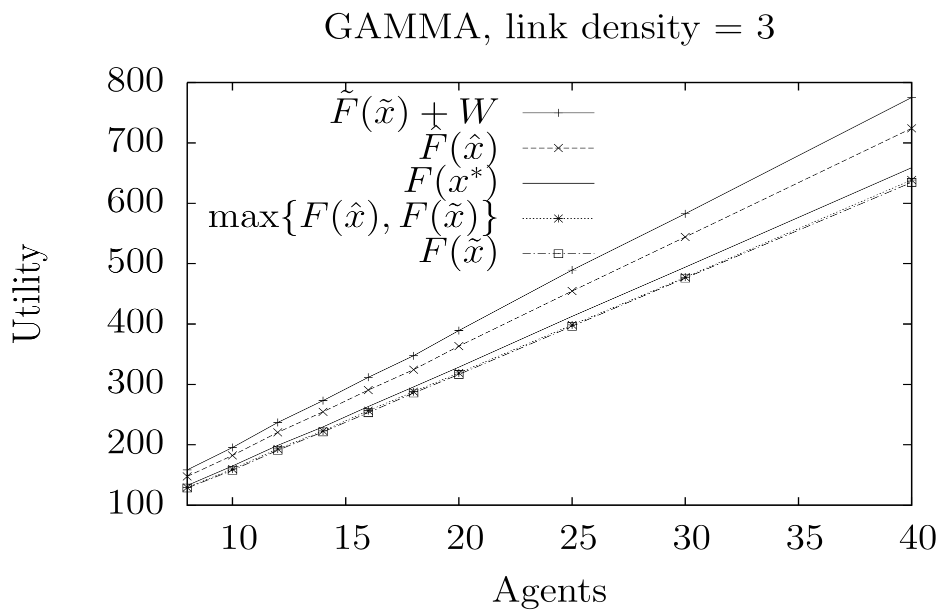
# Experiments

- Evaluate the improvement of:
  - the upper bound of IBMS.
  - the approximation ratio of IBMS.

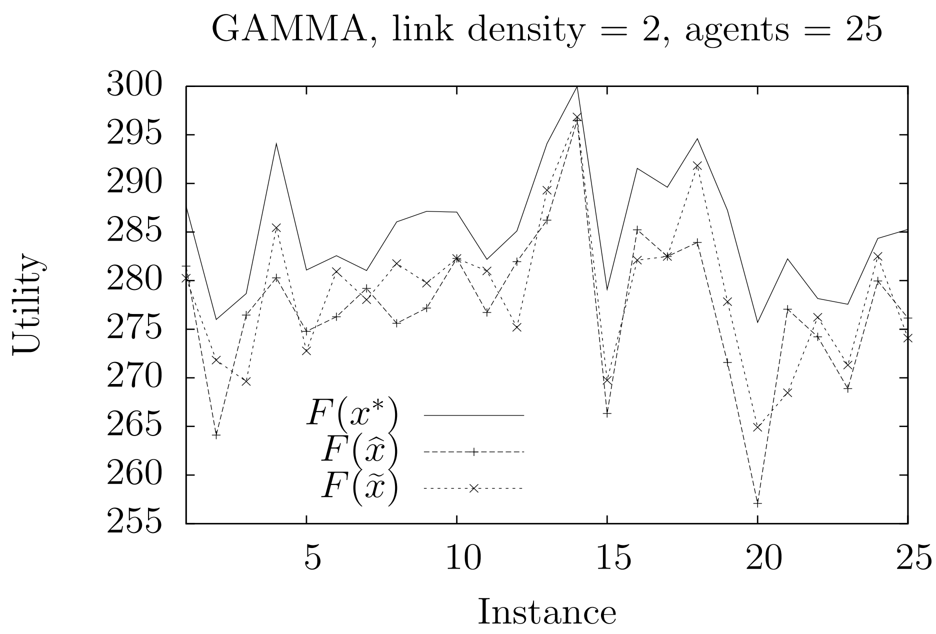
- Graph coloring problems from the ADOPT repository:
  - Two different cost distributions:
    - gamma ( $\alpha = 2, \beta = 3$)
    - uniform
  - Number of variables: [8,…,40]
  - Mean values over 25 repetitions

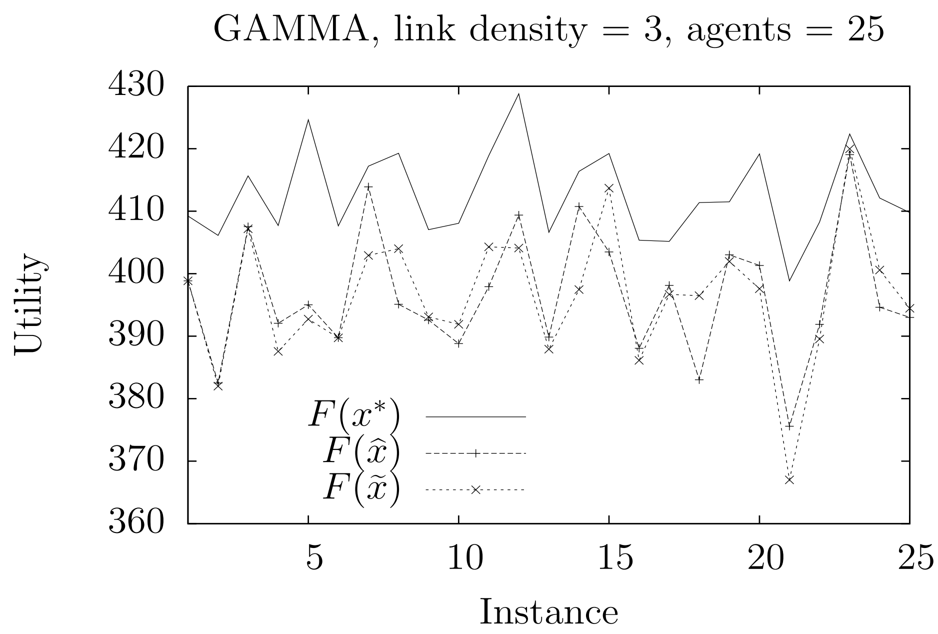
# Upper and Lower Bounds



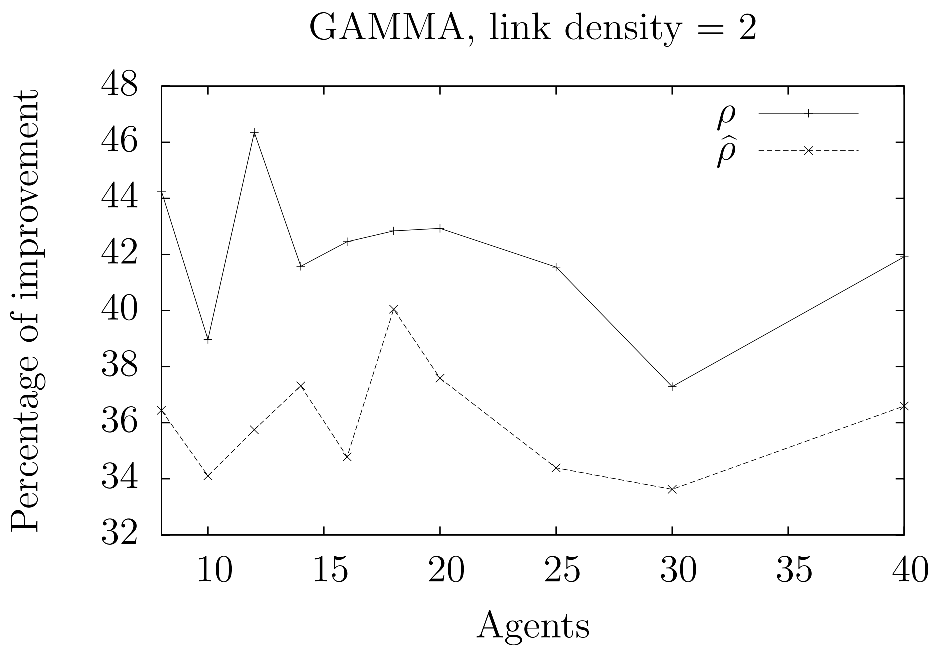



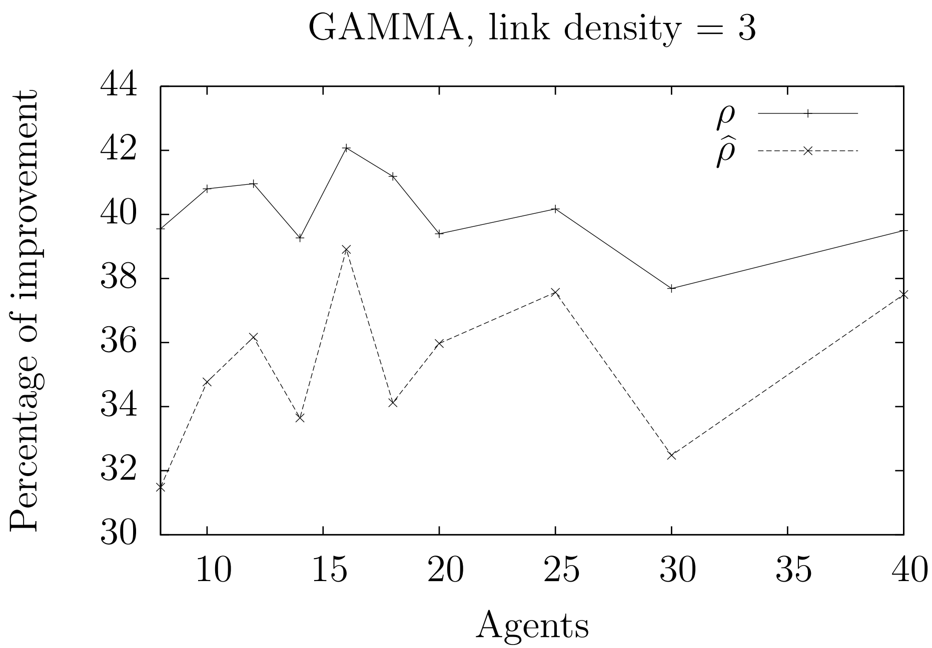
# Detail on the Lower Bounds



# Approximation ratios wrt BMS



# Conclusions

- We introduced IBMS:
  - proved its superiority wrt BMS
  - at the only cost of doubling its communication requirements
- We also introduced weak IBMS:
  - proved its better UB wrt BMS
  - maintain the communication requirements

- Future work: study other relaxation behaviour.

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