Constraint networks

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Constraint Network - Minimal definition

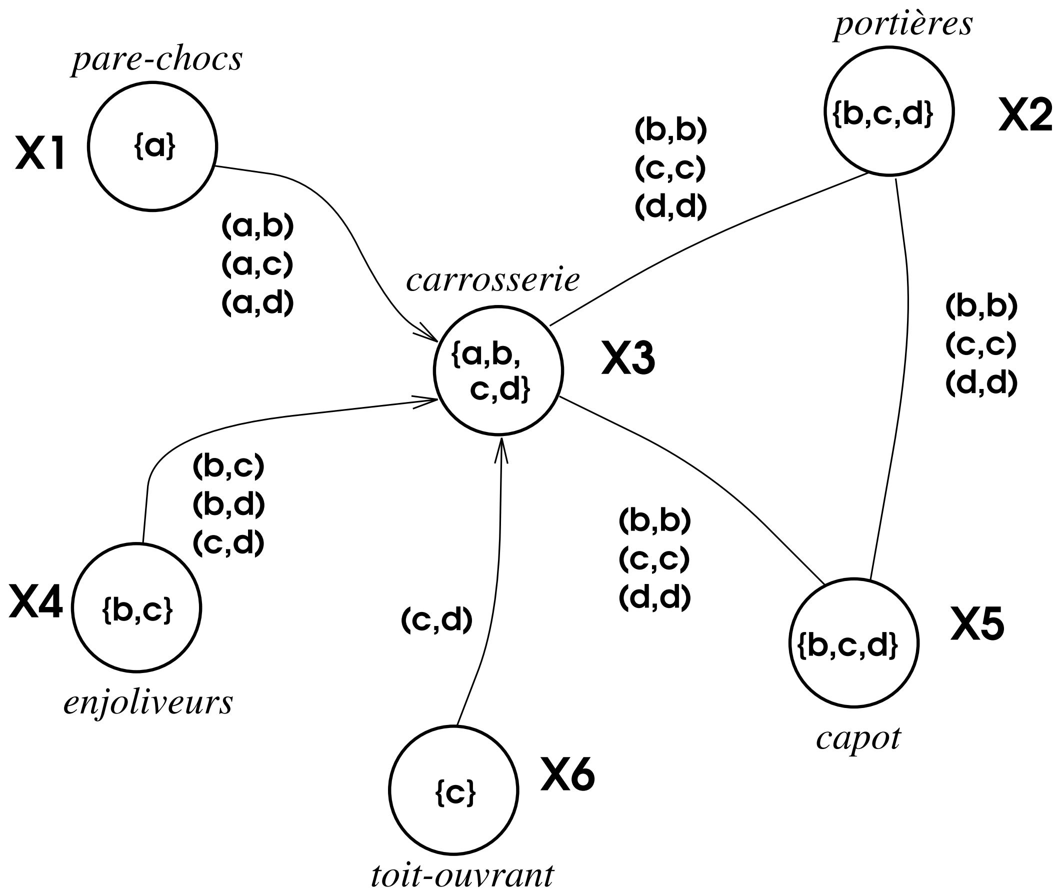
A constraint network (X, D, C):

- $X = \{1, \ldots n\}$ is a set of **variables**.
- $D = \{d_1, \ldots d_n\}$ is a set of finite **domains**.
- $C = \{c_1, \ldots, c_e\}$ is a set of **constraints**.

Each constraint $c \in C$ involves some variables in X and restricts the combination of values these variables may take to a given set of tuples.

The **Constraint Satisfaction Problem**: find an assignment of values to variables such that no forbidden combination of value is used. The decision problem is NP-complete.

Constraint Networks -Example



Constraint networks applications: vision, natural language processing, sequencing, timetabling, assignment, constraint programming. . .

Main limitations

Tentative applications of the CSP technology to real problems raise several issues:

- **Efficiency**: algorithms for the CSP are **time exponential**.
- **Immutability**: the set of constraints is supposed to be **fixed**.
- **Rigidity**: **all** constraints must be **completely** satisfied.

Central motivation: trying to address/understand these limitations.

Arc consistency

One of the simplest and most widely used level of local consistency.

A binary CSP is **arc consistent** if any **single value** assignment can be extended to **any other variable** without any constraint **violation** (viable value)

Arc consistency enforcing: remove non viable values until quiescence.

- massive inference of **unary** constraints preserving equivalence.
- optimal algorithms: AC4/AC6/AC7 $O(ed)$ space and $O(ed^2)$ time.
- may be used as preprocessing or during backtrack tree search (MAC algorithm (Sabin and Freuder 1994; Bessière *et al.* 1995)) as inconsistency detector.

Lazy arc consistency (Schiex *et al.* 1996)

Central idea: instead of producing the complete arc consistent closure, just prove that it is **non empty** (Lazy FC (Zweben and Eskey 1989)).

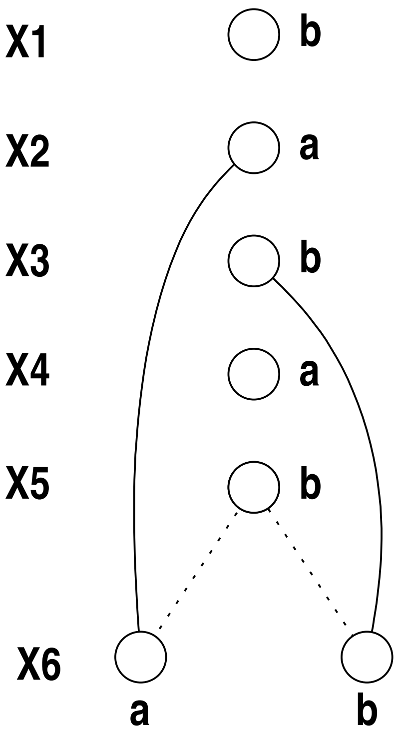
The Lazy AC7 algorithm produced:

- keeps the $O(ed)$ space/$O(ed^2)$ time complexities of AC7
- may be orders of magnitude **faster** and only marginaly slower
- provides extreme **incrementality**
- is still able to identify the **maximum cardinality** arc consistent domain (variale ordering heuristics).

To do: integrate inside Backtrack-tree search (MLAC).

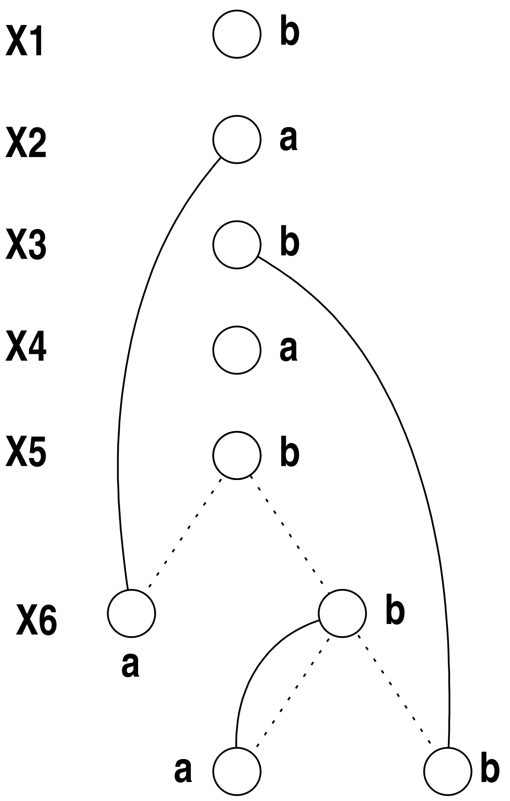
Filtering from failure

During backtrack tree-search, there are situations where one may filter for free.



- **Shallow learning** (Dechter 1990).
- Potential **exponential** number of nogoods learned ⇒ limitation.
- **Backjumping** (BJ): fire and forget nogoods (Gaschnig 1979).

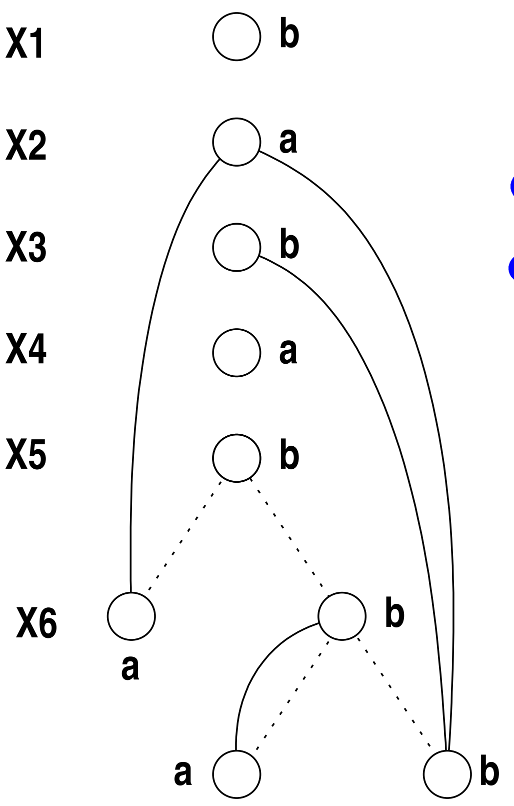
Filtering more: Nogood Recording (Schiex and Verfaillie 1993)



- **NR algorithm**: more filtering, more efficient.
- performs (CBJ) backjumping (Prosser 1993).
- **NR-FC**: extended to work with FC filtering
- **DBT-FC** (Verfaillie and Schiex 1994b): similar ideas applied to DBT (Ginsberg 1993). DBT-MAC is promising (Jussien *et al.* 2000).

Filtering even more: Stubborn NR (Schiex and Verfaillie 1994)

We only build nogoods from **failures**. We should try to fail more.



Central idea: stop constraint checking at the **second** (not first) failure.

- different possible backjumping points (High/Low)
- better efficiency than NR-FC on Prosser's Zebra benchmark
- **puzzling results**. More tests needed

Dynamic CSP

Typical real problems are **not fixed**: constraints/variables/values appear and disappear as time goes. Need for **incremental algorithms**.

Existing work:

- **local consistency**: DnAC4 (Bessière 1991).
- **solution maintenance**: oracles (van Hentenryck 1990).

Two aims:

- **efficient** behavior on **incremental** modifications
- minor solution disturbance on incremental modifications: **optimization**.

Dynamic CSP

Central ideas: reuse previous **solution** (conserved on **relaxation**) or previous **filtering** (conserved on **restrictions**).

- reuse of the previous solution: **Local changes** (Verfaillie and Schiex 1994a).
- reuse of the previous filtering: **justified nogoods** in NR(FC) (Schiex and Verfaillie 1993) and DBT-FC (Verfaillie and Schiex 1994b).

Important improvements. Local changes well adapted to under constrained problems, NR(-FC) to over and critically constrained ones, DBT-FC to both.

Modeling possible states

Not all states are possible. We may therefore use:

- a **CSP** to model the set of all **possible states**
- a **CSP** to model the **problem** to solve
- **relate** the two by constraints.

This defines a **mixed constraint network** (Fargier *et al.* 1996).

We can also use probability distributions on states (Fargier *et al.* 1995).

Modeling possible states

Possible problems:

- **consistency**: **for all** possible states, there **exists** a solution.
- **strong consistency**: there **exist** a solution that works **for all** possible states

Complexity at the **second level** of the PH (Fargier *et al.* 1996).

Incremental algorithms defined.

Probabilistic mixed CSP only NP-complete under independence assumptions (Fargier *et al.* 1995).

CSP and optimization

Real constraint based problems include:

- **hard** constraints
- **preferably satisfied** constraints
- **uncertain** constraints

Modeling such problems as CSP can lead to inconsistency (**over-constrained**) or to abundancy of poor quality solutions (**under-constrained**).

The ability of expressing soft constraints allows one to:

- handle both **over/under-constrained** problems
- define an homogeneous framework for **optimization in CSP**

Existing stuff

- **Max** CSP: minimize the sum of the weights of violated constraints (Shapiro and Haralick 1981; Freuder and Wallace 1992).
- **possibilistic/fuzzy** CSP: minimize the priority of the most important violated constraint (Rosenfeld *et al.* 1976; Schiex 1992).
- **lexicographic** CSP: minimize the number of violated constraint at each evel of priority from the most important to the least important ones (Fargier *et al.* 1993).
- **probabilistic** CSP: minimize the probability of not being a solution of the real problem (Fargier and Lang 1993).

Puzzling situation

In each extension, the usual CSP algorithms, theorems, properties and polynomial classes where (possibly) extended.

Large **redundancy** of work.

- **possibilistic/fuzzy CSP**: (almost) all usual algorithms and properties extend (Rosenfeld *et al.* 1976; Snow and Freuder 1990; Schiex 1992; Fargier 1994; Ruttkay 1994)
- **other extensions**: the majority of the usual properties and algorithms resist to extension. Problems are also harder to solve.

What is the source of this hardness ? Is it possible to bypass it ?

Main motivations

- **Generality**: to build a generic framework that covers existing proposals.
  - avoid redundant work.
  - analyze the differences of behavior.
- **Specificity**: to gather enough properties to be able to built non trivial theorems and generic algorithms.

Eventually: build an homogeneous framework for optimization with constraint networks.

The VCSP Framework (Schiex *et al.* 1995)

- with each **constraint** we associate a **valuation** that reflects its importance: preference, weight, priority, probability of existence. . .
- The valuation of an assignment is defined as the **combination** of the valuations of the **violated constraints** using an operator that must satisfy a set of axioms.
- The instanciations can be **compared** using a **total order** on valuations.
- The problem is to produce an assignment of **minimum valuation**.

Commutative totally ordered semigroup with a monotonic operator.

Similar approaches

- **(Minoux 1976)**, **Aho/Ullman** 197? : **semi-ring** used to generalize all-pair shortest path algorithms;
- **(Shafer and Shenoy 1988)** : a **semi-group** structure is used to generalize dynamic programming like algorithms (variable elimination, CSP: adaptative consistency, tree-clustering. . .);
- Reasoning under uncertainty: **triangular norms and co-norms** are based on semi-groups.
- **Partial CSP** (Freuder 1989), uses a notion of **distance** between problems.
- **(Bistarelli et al. 1995)** : a **semi-ring** structure is used to extend CSP, dynamic programming-like algorithms and local consistency algorithms;

Similar structures, different aims, different compromises.

More precisely

- $\mathrm{VCSP} = (V, D, C, S, \varphi)$, $S = (E, \succeq, \top, \bot, \oplus)$
- E = **set of valuations**, made of numbers or symbols, used to annotate constraints and to assess assignments;
- $\top$ = **maximum element** of E, used to annotate hard constraints, corresponds to totally inconsistent assignments;
- $\bot$ = **minimum element** of E, corresponds to completely consistent assignments;
- $\succeq$ = **total order** on E, used to compare two valuations;
- $\oplus$ = **operator** used to **combine** two valuations;
- $\varphi : C \rightarrow E$ = **valuation fonction**, used to annotate each constraint with a valuation.

Different sub-classes

CSP	E	$\succ$	$\top$	$\perp$	$\oplus$
classical	{t,f}	f $\succ$ t	f	t	$\wedge$
Max-CSP	$\overline{\mathbb{N}}$	$>$	$+\infty$	0	$+$
possibilistic	$[0, 1]$	$>$	1	0	max
lexicographic	$[0, 1]^*$	$>^*$	$\top$	$\emptyset$	$\cup$
probabilistic	$[0, 1]$	$>$	1	0	$a \oplus b = 1 - (1 - a)(1 - b)$

Two important additional properties

- **idempotency**: $\forall a \in E, a \oplus a = a$
- **strict monotonicity**: $\forall a, b, c \in E, a \succ b \wedge c \neq \top \Rightarrow (a \oplus c) \succ (b \oplus c)$

Using VCSP axioms, we have:

- $|E| > 2 \Rightarrow \neg($**idempotency** $\wedge$ **strict monotonicity**$)$.
- **idempotency** $\Rightarrow \oplus = \max$. Drowning effect. Local consistency.

  $A_1 \quad : \{0.6\}$ $\qquad\qquad$ $A_2 \quad : \{0.6, 0.5, 0.4, 0.4, 0.4\}$

- **strict monotony**: no drowning effect/local consistency.

Max-CSP and guaranteed optimality

- **Dynamic programming**: applies directly to **acyclic** networks $\Rightarrow$ **Polynomial class** of well-behaved (bounded induced width) graphs (Shenoy 1991; Dechter *et al.* 1990; Bistarelli *et al.* 1995).
- **Branch and bound**:
  - systematic search using a **depth first** strategy.
  - **upper bound** $\mathfrak{ub}$: valuation of the best known solution.
  - **lower bound** $\mathfrak{lb}$: on the best valuation one could obtain by extending the current assignment to a complete one.
  - **backtrack**: when $\mathfrak{lb} \succeq \mathfrak{ub}$.

Max-CSP & lower bounds

- **Backward checking** (BC): exploits constraints between assigned variables.
- **Forward checking** (FC): exploits also constraints between assigned and unassigned variables.
- Constraints among unassigned variables:

  Arc consistency !

  First trial, using the notion of relaxation:

  MAX-AC-VCSP is **NP-complete** in stricly monotonic structures (Schiex *et al.* 1995)

Max-CSP: improved lower bounds

- **Directed arc-consistency counts** (DAC) (Wallace 1994): static variable ordering. Constraints propagated in one direction fixed by the variable ordering.
- **Russian dolls** (Verfaillie *et al.* 1996): uses the optimal valuation of the problem defined by unassigned variables combined with the (BC+FC) lower bound, inductively.
- **Reversible DAC** (Larrosa *et al.* 1998; 1999): dynamic variable ordering. Each constraint is propagated in a direction that (cleverly) changes during search.
- **Weighted arc-consistency counts** (WAC) (Affane and Bennaceur 1998): each constraint is propagated partly in one direction, the rest in the other. Generalizes RDAC.

The taste of local consistency without local consistency !

Max-CSP: arc consistency (Schiex 2000)

AC can be extended to **Max-CSP** by:

- clearly **separating** information flow (constraints to domains/domains to constraints).
- preserving **equivalence** by **compensating** for induced unary constraints (using substraction).
- **loss**: uniqueness of the AC closure. Each closure induces a LB.
- this LB can be at least **twice larger** than RDAC/WAC and may be infinitely stronger.

Similar work: Penalty shifting (Koster 1999), probabilistic AC (CP'2000). Several open questions.

Frequency assignment (Cabon *et al.* 1999)

Given a set of radio **communication links**, assign to each link two frequencies such that:

- they satisfy a **fixed distance** constraint between the two frequencies of a same link

$$|f_i - f_{i+1}| = K$$

- theu minimize the **interferences** (weighted sum of violated constraints).

$$|f_i - f_j| > \alpha_{ij} \qquad (w_{ij})$$

Optimality proof obtained by solving a sub-problem SUB_4 of CELAR6 using a parallelized improved **RDS** implementation (5.10^6 seconds of Sparc 5) (de Givry *et al.* 1997).

Other results

Using the **reversible DAC** based algorithms (Sparc 2).

	PFC-DAC		PFC-RDAC		PFC-MRDAC	
	nodes	cpu	nodes	cpu	nodes	cpu
SUB_1	8.1×10^8	5.2×10^4	1.9×10^7	3.5×10^3	1.8×10^6	2.6×10^3
SUB_2	4.1×10^9	2.5×10^5	1.2×10^8	2.9×10^4	1.6×10^7	2.3×10^4
SUB_3	$*$	$*$	2.7×10^8	7.1×10^4	3.7×10^7	6.8×10^4
SUB_4	$*$	$*$	1.2×10^9	3.4×10^5	1.2×10^8	2.6×10^5

Since, CELAR 6, 9 and 10 have been directly solved using **dynamic programming** and a nice tree decomposition heuristics (Koster 1999).

Main research directions

- **practical** exploitation of graph structure in **dynamic programming** and **tree search** algorithms beyond acyclic problems (treewidth,cutset number, pseudo-tree height)...

- improving **lower bounding functions** for **branch and bound methods**.

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