Weighted CP & related framemorks

Soft local consistencies as bounds

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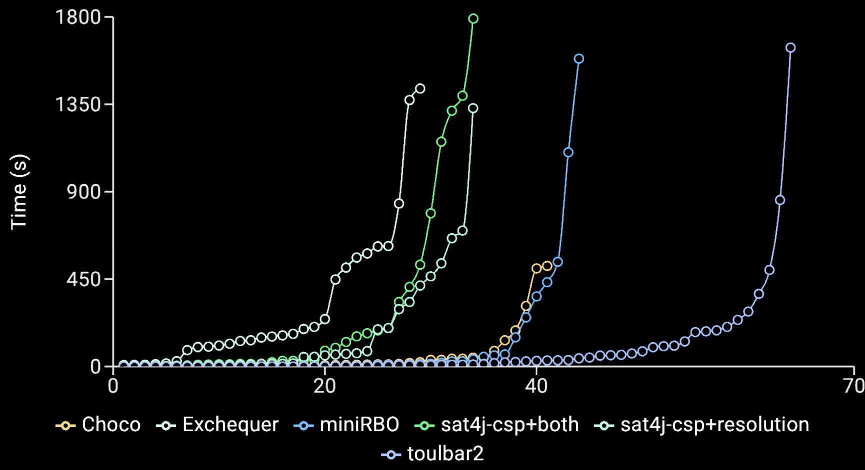
XCSP3 MiniCOP track

Problems solved with a proof of optimality



A Constraint Network $\langle \boldsymbol{V}, \Phi \rangle$

- a sequence of discrete domain variables $\boldsymbol{V}$
- a set Φ of e Boolean functions (or constraints)
- Each $c_{\boldsymbol{S}} \in \Phi$ is a truth function from $D^{\boldsymbol{S}} \to \{t, f\}$

Joint truth function

$$\Phi_{\mathcal{M}} = \bigwedge_{c_{\boldsymbol{S}} \in \Phi} c_{\boldsymbol{S}}$$

The Constraint Satisfaction Problem (NP-complete)

- Is it possible to have $\Phi_{\mathcal{M}} = t$?

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Languages for domains and constraints

- Constraint Networks: Boolean tables (tensors) for domains and constraints
- SAT/CNF: Boolean variables and clauses
- Constraint Programming: interval domains, global constraints,...

Tables (or tensors) for $c_{\boldsymbol{S}}$

- A multidimensional table with a Boolean for every tuple in $D^{\boldsymbol{S}}$
- Says if it is authorized (t) or not (f)

Pairwise difference (3 values)

$$\begin{bmatrix} f & t & t \\ t & f & t \\ t & t & f \end{bmatrix}$$

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Most famous

AllDifferent*s*

Cost Function Network $\langle \boldsymbol{V}, \Phi, \top \rangle$

- a sequence of discrete domain variables $\boldsymbol{V}$
- a set Φ of e **non negative** integer cost functions
- each $c_S \in \Phi$ is a numerical function bounded by $\top$ — $c_\varnothing$ lower bound

Joint cost function using $a +^\top b = \min(a + b, \top)$ — saturating arithmetics

$$\Phi_{\mathcal{M}} = \sum_{c_S \in \Phi}^{\top} c_S$$

The Weighted Constraint Satisfaction Problem (decision NP-complete)

- What/where is the minimum of $\Phi_{\mathcal{M}}$?

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- A multidimensional table with a number for every tuple in D^S

Beyond dense tensors

- Weighted clauses: a cost for a cell in a (sparse) tensor
- Names for specific (useful) global cost functions

Soft difference (3 values)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

$\textsc{WeightedRegular}_{\mathcal{A},S}$

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$\textsc{WeightedRegular}_{\mathcal{A},S}$

Graph $G = (\boldsymbol{V}, \boldsymbol{E})$ with edge weight function w

- A Boolean variable X_i per vertex $i \in V$
- A cost function per edge $e = (i, j) \in E : c_{ij} = w(i, j) \times \mathbb{1}[x_i \neq x_j]$

A simple graph

- vertices $\{1, 2, 3, 4\}$
- cut weight 1 or 1.5 $(1, 3)$
- edge $(1, 2)$ hard

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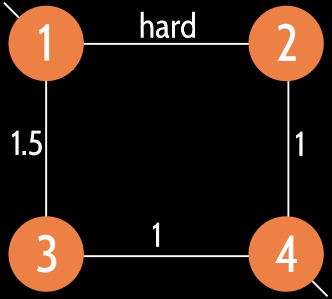


Min-CUT on 4 variables

```
import pytoulbar2
myCFN = pytoulbar2.CFN(100,1)   # T/ub, resolution
for i in range(4):
    myCFN.AddVariable("x"+str(i+1),["l", "r"]) # returns an index
myCFN.AddFunction(["x1"],[0,100])
myCFN.AddFunction(["x4"],[100,0])
myCFN.AddFunction(["x1","x3"], [0,1.5,1.5,0])
...
sol = myCFN.Solve()  # returns a triple (sol, cost, _)
```

Expressing probabilities through cost/energy distributions

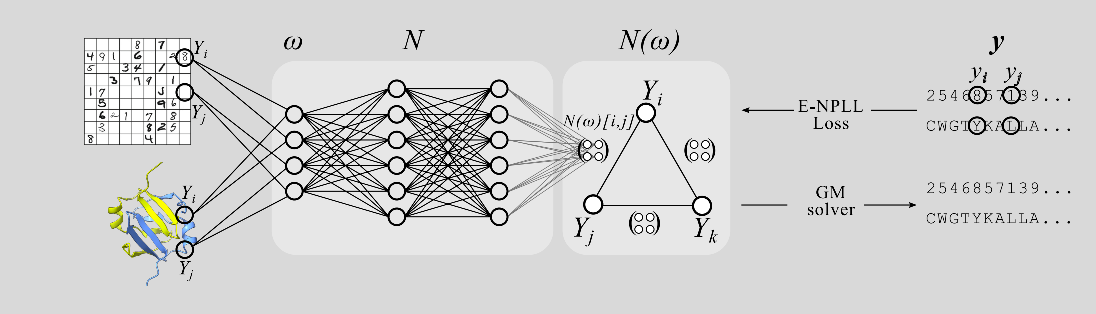
- log-linear models

$$P(t) \propto \exp(-\Phi(t))$$

- Normalization is #P-hard
- Optimisation and counting (marginals)

Markov Random Fields, Bayesian nets, Factor Graphs, Ising/Potts models [12]

- log-linear MRFs $\approx$ CFNs (cost = energy). Image/signal processing.
- CFNs can be learned from data.



This can [5, 9, 10]

- Simultaneously learn how to decode numbers and play Sudoku
- Or learn costs from contextual data and solutions (Decision focused learning)
- Or learn how to design new proteins (CoViD neutralizers)

Uses a 0/1 encoding of finite domains: $x_{ia} = 1$ means $x_i = a$.

Pairwise models [2]

$$\text{Minimize} \sum_{i,a} c_i(a) \cdot x_{ia} + \sum_{\substack{c_{ij} \in \Phi \\ a \in D^i, b \in D^j}} c_{ij}(a,b) \cdot x_{ia} x_{jb} \quad \text{such that}$$

$$\sum_{a \in D^i} x_{ia} = 1 \qquad \forall i \in \{1, \ldots, n\}$$

$$x_{ia} \in \{0, 1\} \qquad \forall i \in \{1, \ldots, n\}$$

nd variables, n constraints: concise but challenging

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The "local polytope" [22, 14, 24, 23] (without eq. (1))

Minimize $\sum_{i,a} c_i(a) \cdot x_{ia} + \sum_{\substack{c_{ij} \in \Phi \\ a \in D^i, b \in D^j}} c_{ij}(a,b) \cdot p_{iajb}$ such that

$$\sum_{a \in D^i} x_{ia} = 1 \qquad \forall i \in \{1, \dots, n\}$$

$$\sum_{b \in D^j} p_{iajb} = x_{ia} \qquad \forall c_{ij} \in \Phi, \forall a \in D^i$$

$$\sum_{a \in D^i} p_{iajb} = x_{jb} \qquad \forall c_{ij} \in \Phi, \forall b \in D^j$$

$$x_{ia} \in \{0, 1\} \qquad \forall i \in \{1, \dots, n\} \qquad (1)$$

$nd + ed^2$ variables, $n + 2ed$ constraints: strong but expensive

A well-studied object

The hope was that local polytopes could be solved faster than LP

Plenty of fast **approximate** Block Coordinate Descent algorithms [20]

Local polytope universality: constructive proof [18]

Any LP can be reduced in linear time to a linear optimization over a local polytope (with infinite weights).

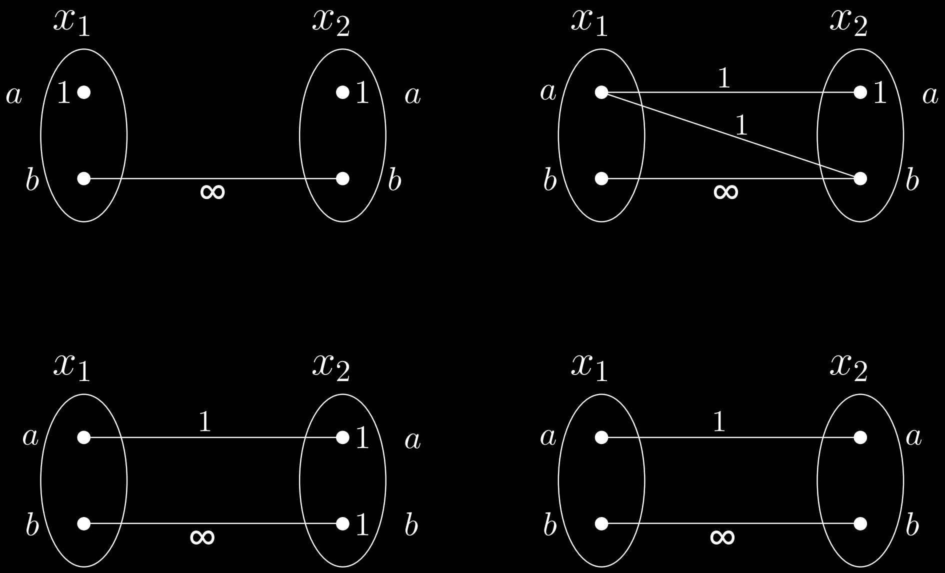
Fundamental for Tree Search efficiency

- Tight bounds
- Efficient bounds

Main difference with CP

- (G)AC: variables and constraints communicate through domains (value deletion)
- Soft (G)AC: variables and cost functions communicate through unary cost functions [21].
- This includes value deletions ($\top$)

Simple example

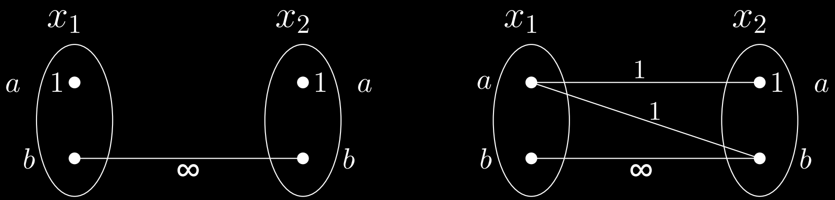


$$c_\varnothing = 1$$

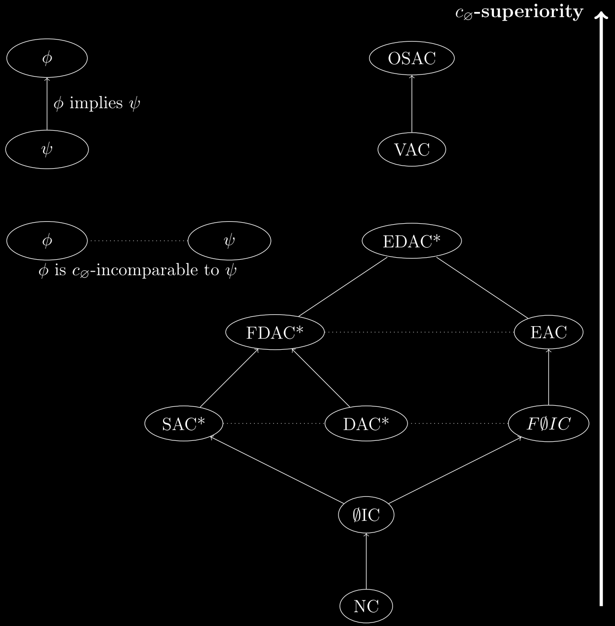
Procedure MoveCost($c_{\boldsymbol{S}}, c_{\boldsymbol{S}'}, \tau, \alpha$): Move α units of cost between the tuple τ of scope $\boldsymbol{S}$ and tuples τ' that extend τ in scope $\boldsymbol{S}'$

Data: Scopes $\boldsymbol{S} \subset \boldsymbol{S}'$
Data: $\tau \in \ell(\boldsymbol{S})$
Data: cost α to move
$c_{\boldsymbol{S}}(\tau) \leftarrow c_{\boldsymbol{S}}(\tau) + \alpha$;
foreach $\tau' \in \ell(\boldsymbol{S}') \mid \tau'_{\boldsymbol{S}} = \tau$ **do**
  $c_{\boldsymbol{S}'}(\tau') \leftarrow c_{\boldsymbol{S}'}(\tau') - \alpha$;

$\mathbf{S} = \{x_1\} \quad \mathbf{S}' = \{x_1, x_2\} \quad \tau = (a) \quad \alpha = -1$



Fix-point algorithms using a well-defined set of Move operations



Algorithm	Time Complexity
NC	$O(nd)$
$\emptyset$IC	$O(ed^2)$
$F\emptyset IC$	$O(ed^2)$
SAC	$O(n^2d^2 + ed^3)$
DAC	$O(ed^2)$
FDAC	$O(end^3)$
EDAC	$O(ed^2max(nd, \top))$
VAC_ε	$O(ed^2\top/\varepsilon)$
OSAC	$O(e^{4.5}d^{5.5}logM)$

Generalize usual (G)AC

- Enforce GAC when applied to constraints
- Most are polytime (some depend on $\top$)
- Produce an equivalent problem (incremental)
- Can have several fixpoints

Virtual Arc Consistency [7]

Definition

A CFN is VAC iff its hardened version has a non empty arc consistent closure.

Hardened: non zero-costs are forbidden ($\top$).

Enforcing VAC — AC-core guided

- Iteratively enforces AC on the hardened version to build a sequence of moves that increase $c_\varnothing$.
- Generalization of Ford Fulkerson (Augmenting DAGs [15])
- With capacity scaling (MaxSAT stratification)
- Solves submodular problems (Min-CUT)

What "Move" operations are used in enforcing Soft AC

- Moving to the 0-ary scope from a unary scope y_i
- Moving between a value a's cost and associated pairs in c_{ij} y_{iaj}
- We should not create negative costs (lower bound)

$$\max \left(c_{\varnothing} + \sum_{i \in \{1,\ldots,n\}} y_i \right)$$

such that:

$$c_i(a) - y_i + \sum_{j \in \{1,\ldots,n\}} y_{iaj} \geq 0 \qquad \forall i \in \{1, \ldots, n\}, \forall a \in D_i$$

$$c_{ij}(ab) - y_{iaj} - y_{jbi} \geq 0 \qquad \forall (i,j), i > j, \forall (a,b) \in D_i \times D_j$$

OSAC is the dual of the local polytope

Soft Arc Consistencies are

- Fast combinatorial heuristics that find good feasible dual solutions
- The reduced costs are used in the reformulated problem
- The problem remains equivalent

VAC vs. LP on Protein design problems

CPLEX V12.4.0.0

```
Problem '3e4h.LP' read.
Root relaxation solution time =  811.28 sec.
...
MIP - Integer optimal solution:  Objective =  150023297067
Solution time =  864.39 sec.
```

tb2 and VAC (AC3 based)

```
loading CFN file: 3e4h.wcsp
Lb after VAC: 150023297067
Preprocessing time: 9.13 seconds.
Optimum: 150023297067 in 129 backtracks, 129 nodes and 9.38 seconds.
```

Could this be useful for ILP?

By reversing the universal polytope construction somehow?

Let P be an LP $\{\min c^T z | Az = b, z \geq 0\}$

$\mathbf{y}$ *a dual solution of P with cost* $\mathbf{y}^T b$

associated reduced costs $rc^{\mathbf{y}} = c - A^T \mathbf{y}$.

. *Then the LP* $\{\min rc^{\mathbf{y}^T} z + \mathbf{y}^T b | Az = b, z \geq 0\}$ *is equivalent to P.*

The ingredients

- An All-different constraint
- Unary cost functions on all involved variables

Reformulate this subproblem to maximally increase the lower bound $c_\varnothing$

How to reformulate?

- Find an optimal dual solution of this Linear Assignment Problem
- Reformulate using the dual solution

The Primal

$$\min \sum_{i,j \in [1,n]} c_i[j] x_{ij} \quad s.t.$$

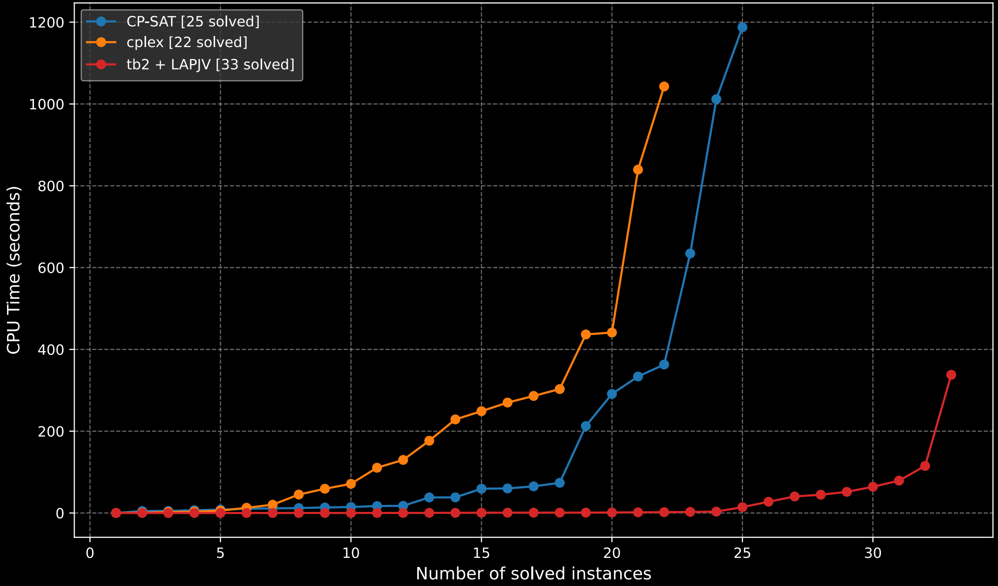
$$\sum_{j \in [1,n]} x_{ij} = 1 \qquad \forall i \in [1,n]$$

$$\sum_{i \in [1,n]} x_{ij} = 1 \qquad \forall j \in [1,n]$$

We can

- solve it using a primal-dual algorithm such as LAPJV (Jonker & Volgenant [13]) algorithm.
- Preserve primal/dual solutions during tree search for incrementality

Quadratic Assignment Problems (QAPLib) $n \leq 100$



We only skimmed through the surface — **Soft Local consistencies**

- On-the-fly variable elimination [16]
- Parallel Hybrid BFS [1, 3]
- Tree decomposition-aware tree-search ÷ [11, 19, 1] (closed several RLFAP instances)
- Dominance analysis [8, 2]
- A parallel complete VNS-LDS-WCP local search algorithm [6, 17]...

Visit toulbar2 websites: `github.com/toulbar2/toulbar2` and `toulbar2.github.io`

Thank you all for your attention!

And to all CFN/toulbar2 contributors

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Questions?

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