Semiring-Based and Valued CSPs Revisited

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Abstract

An a posteriori revisit of the accumulated results on the Semiring and Valued constraint satisfaction problems and related problems, showing how semi-rings and monoids (or semi-groups) are useful objects to explore the intricate interplay between expressivity and the design of efficient, correct and convergent algorithms in computer science, showing tight relationships and scientific overlaps with probabilistic reasoning over graphical structures and linear or polynomial discrete mathematical programming.

Keywords: Semiring, Valuation structures, Constraint Satisfaction, Soft Constraints, Monoids, Tropical algebra, Graphical Models, Complexity, Arc Consistency, Variable Elimination, Non-serial Dynamic Programming, Probabilistic Reasoning

1 Introduction

Constraint Networks (CNs) have long provided a unifying framework for modeling combinatorial problems in artificial intelligence. Discrete variables, with finite domains, are used to represent the discrete features of a given problem and constraints involving a subset of these variables are defined by sets of authorized/forbidden combinations. The usual problem is then to assign all variables with a value of their domain so that only authorized combinations of values are used. This is called the Constraint

Satisfaction Problem (CSP), a NP-hard problem that gave birth to Constraint Programming [1]. However, classical CNs are inherently limited to *crisp* constraints, where tuples are either allowed or forbidden.

In many real-world applications, this binary view proved insufficient. Problems in areas such as decision-making [2], resource allocation [3], diagnosis [4], and planning [5] naturally involve preferences, costs, uncertainty, or partial satisfaction. Early attempts to extend CNs led to a variety of frameworks, including fuzzy CNs [6, 7], possibilistic CNs [8], probabilistic [6, 9], lexicographic [10, 11] an weighted CNs [12] with an associated notion of variable assignment quality. While successful in specific domains, these approaches remained largely disconnected, each tailored to a particular semantics and requiring specific theoretical analysis, algorithmic developments and codes.

The Semiring and Valued constraint networks were introduced to address this fragmentation by providing unifying algebraic structures capable of capturing a wide spectrum of soft constraints within general algebraic frameworks [11, 13, 14] and compared in the original paper discussed here [15]. The key idea was to abstract the notion of constraint satisfaction into a valuation-based setting, where constraints produce values that can be combined and compared according to a suitable algebraic structure. This shift from feasibility to valuation marked a conceptual transition: constraints were no longer simply Boolean functions, but more expressive functions contributing to a global measure of solution quality.

2 The Semiring-Based Framework

The Semiring constraint satisfaction framework relies on an algebraic structure $(A, \oplus, \otimes, \mathbf{0}, \mathbf{1})$ called a c-semiring, where values are drawn from a set A, combined using a multiplicative operation $\otimes$, and *compared* via an additive operation $\oplus$ [13, 14]. Beyond the usual commutative semi-ring axioms ($\otimes$ and $\oplus$ being associative and commutative with $\otimes$ distributing over $\oplus$, $\mathbf{1}$ being the identity for $\otimes$ and $\mathbf{0}$ the identity of $\oplus$ and the absorbing element of $\otimes$), the c-semiring structure assumes that $\oplus$ is idempotent and that $\mathbf{1}$ is the absorbing element for $\oplus$. These 2 extra properties restrict $\oplus$ to define a partial order over A defined as $a \leq b$ iff $a \oplus b = a$: $\oplus$ is the least upper bound operator in a lattice anchored in $\mathbf{0}$ and $\mathbf{1}$. This gears Semiring constraint satisfaction framework towards maximization problems.

In this setting, soft constraints are functions that assign values to tuples of variable assignments. The minimum valuation $\mathbf{1}$ is associated with complete satisfaction while $\mathbf{0}$ is associated with complete violation: a classical constraint can be directly represented as a function using only $\{\mathbf{0}, \mathbf{1}\}$ as valuation degrees. These values are combined across soft constraints using $\otimes$, and the quality of different assignments is compared using $\oplus$. The problem of identifying an optimal solution, one that has the best value according to the ordering induced by $\oplus$ is the Semiring Constraint Satisfaction Problem, or SCSP, capturing a wide variety of problems, including:

- the classical CSP (as a special case), using $\otimes = \wedge$ and $\oplus = \vee$ over Booleans,
- the fuzzy CSP, using $\otimes = \min$ and $\oplus = \max$ over the $[0, 1]$ interval.
- the possibilistic CSP, using $\otimes = \max$ and $\oplus = \min$ over the $[0, 1]$ interval.
- the probabilistic CSP, using $\otimes = \times$ and $\oplus = \max$ over the $[0, 1]$ interval.

- the weighted CSP, using $\otimes = +$ and $\oplus = \max$ over the $\mathbb{N} \cup \{\infty\}$.

One of the main contributions of the c-semiring framework is thus its ability to serve as a *unifying language* for optimization over soft constraints, separating the structure of the problem from the semantics of preferences.

A central theme in the development of soft constraint frameworks is the trade-off between generality and the capacity to build effective algorithms and interesting results without too many extra assumptions. The SCSP framework offers a high degree of generality, allowing arbitrary c-semirings to model a wide range of valuation systems. However, this flexibility comes at a cost: fewer structural assumptions imply weaker guarantees for efficient or correct algorithmic solutions.

The Valued Constraint Satisfaction (VCSP) framework adopts a more structured approach geared at minimization problems. Very concisely, the VCSP framework is the SCSP framework with the extra assumption that the valuation set A is totally ordered, using $\oplus = \min$. Instead of depending on the two operators of a c-semiring, a single operator of a totally ordered semigroup (or monoid) is used, defining a more constrained structure that facilitates the development of algorithmic techniques and theoretical results.

Rather than viewing these frameworks as competing, it is more appropriate to see them as complementary: the SCSP framework provides a general and expressive modeling language, while the VCSP framework exploits its tighter structure to reach stronger algorithmic and theoretical characterizations [15]. This comes at a cost however, as c-semirings allow for more general valuation structures, including partially ordered valuation domains, which cannot be directly captured in the Valued framework, by definition.

Understanding the precise connections between these frameworks has contributed to a deeper insight into the nature of soft constraints, clarifying how different modeling choices impact both expressiveness and computational properties.

3 Theoretical and Algorithmic Insights

One of the lasting contributions of the Semiring and Valued approaches is the introduction of an algebraic view of constraint satisfaction. By separating the combinatorial structure of the problem from the semantics of valuations, SCSPs provide a modular framework in which different notions of optimality can be studied independently of the underlying constraint network. This perspective has influenced subsequent work in both constraint programming and related areas, emphasizing the importance of abstract structures in understanding computational properties.

One of the first property of the Semiring constraint satisfaction framework is the existence of a general variable elimination scheme for solving arbitrary SCSPs [13]. As we will see later, this generic algorithm was already known in the early 90s. Variable elimination, also known as non-serial dynamic programming [16], is theoretically important, being related to resolution in logic [17, 18]. As it, it is plagued with worst-case exponential space/time complexities which reduce its applications to specific sub-classes of problems with low treewidth [19].

Beyond variable elimination, the practically useful and important notion of "arc consistency" in classical CSPs was immediately targeted for generalization. A constraint network is said to be arc-consistent if any feasible assignment of one variable (one not explicitly forbidden by a constraint involving this variable alone), can be extended to a feasible assignment over any extra variable (one not forbidden by a constraint involving this extra variable). Constraint networks can be filtered by arc consistency: an initial network is transformed into an equivalent one (that has the same set of solutions) that is also arc consistent. The resulting problem is unique, simpler, and can be computed by efficient *arc consistency enforcing algorithms* [1]. These algorithms are at the core of all constraint programming tools as they allow for the efficient detection of some infeasible problems, when a variable domain becomes empty. In most existing CP solvers, arc consistency is incrementally maintained at each node during backtrack-search to prune a Depth-First search tree.

In the c-semiring setting, arc consistency was generalized to a soft counterpart [14]. This generalization preserves the intuition of filtering inconsistent or suboptimal values, but account for the more complex aggregation of valuations. However, this generalized filtering algorithm, obtained by replacing $\wedge$ by $\otimes$ and $\vee$ by $\oplus$ in the classical arc consistency enforcing algorithm, works only when $\otimes$ is idempotent ($a \otimes a = a$ for any valuation a). This is the case for the max and $\wedge$ operators for example. In these cases, the resulting $\max - \min$ SCSPs consist in maximizing the least satisfied constraint in a network and generalizes to partially ordered structures such as those defined by multi-criteria fuzzy constraint networks, a situation for which no algorithm existed.

However, for non-idempotent operators such as $\times$ or $+$, the resulting algorithm may fail to converge, and may produce a non-equivalent problem, making it useless in practice for such operators. As a matter of fact, probabilistic reasoning would require using $\otimes = \times$, and the $(\mathbb{R} \cup \{\infty\}, \min, +)$ semi-ring, also known as the min-tropical semiring [20], is useful to represent additive costs and constraints (using ∞). This frustrating situation was thoroughly analyzed in the Valued framework, with the aim of producing *functional* arc consistency properties and enforcing algorithms that would work with both idempotent and non-idempotent $\otimes$ operators, providing polynomial time local consistency enforcing, production of an equivalent problem and of an associated bound on the optimal valuation of the problem that could be incrementally enforced for pruning during Branch and Bound search.

For VCSPs alone, a first definition of arc consistency for $\otimes = +$ had been proposed in [11], but it was not polytime nor incremental. Eventually, the first *functional* definition of soft arc consistency for VCSPs, generalizing classical arc consistency to both idempotent and non-idempotent Valued constraint networks, was introduced in [21]. It however requires a specific extra-property of the operator $\otimes$ called "fairness". For a valuation structure to be fair, the $\otimes$ operator must have a pseudo-inverse operator, $\oslash$ such that $(a \otimes b) \oslash b = a$. This inverse operator offers some of the properties of a full algebraic group: if a valuation b is combined to a given valuation a, it is possible to cancel this combination. In a group, this would be feasible by combining $(a \otimes b)$ with the inverse of b. In a fair semi-group, this is achieved instead by using the pseudo-inverse operator and b itself. This operator allows for the definition of "Equivalence

Preserving Transformations" (EPTs) on fair Valued constraint networks, allowing to shift costs between functions, leading to polytime soft arc consistency enforcing algorithm producing an equivalent problem for $\otimes = \times$ or $+$. These EPTs shift specific amounts of valuations between soft constraints while preserving equivalence. In the resulting problem, a constant soft constraint (involving no variable) directly provides a bound that can be used to prune during search.

The algorithmic significance of "fair" Valued constraint networks has motivated an exact mathematical characterization of all fair valuation structures: it has been shown that such structures are formed from a fully ordered stack of valuation sets, combinations of elements inside stacks working as in additive problems ($\otimes = +$) while combination between the extreme elements of these stacks act as in idempotent min-max problems [22]. Since functional algorithms for enforcing arc consistency had previously been proposed for the idempotent $\otimes$ cases [7, 8, 14], this result helped in focusing research on the specific crucial case of additive valuations. The specific instance of valuation structure, using $\otimes = +$, $\oplus = \min$ and valuations in $\mathbb{N} \cup \{\infty\}$ is usually called a Cost Function Network (CFN [23]) and the problem of finding a variable assignment that minimizes the sum of all cost functions is the Weighted Constraint Satisfaction Problem (WCSP), also related to Partial Constraint Satisfaction [12].

When enforcing arc consistency on a CFN, several equivalent arc consistent problems may exist, some defining tighter bounds than others. This multiplicity of arc consistent closures has given raise to a long series of refined soft arc consistency properties and associated enforcing algorithms [23], each trying to trade bound tightness with computational efficiency: arc consistency [21], full and directional arc consistency [24, 25], existential arc consistency [26], virtual arc consistency [27] and optimal soft arc consistency (OSAC [28]). This last variant, as its name indicates, identifies an optimal equivalent sub-problem, but enforcing it requires solving a large linear program (LP) which makes it impractical for branch and bound solving. This connection with LP offers a deep understanding of the nature of all soft arc consistency algorithms: the OSAC LP is the dual of the so-called "local polytope" [29], the continuous relaxation of the natural formulation of the WCSP as an LP over 0/1 variables. Cost shifted in EPTs can then be analyzed as feasible dual solutions of this relaxed LP and soft arc consistency enforcing as heuristics that efficiently provide feasible but possibly suboptimal dual multipliers [23, 28, 30].

Important results on computational complexity and tractable classes for CFNs are reviewed in [19, 31]. Thanks to the availability of variable elimination algorithms, bounded-treewidth problems define a significant polynomial structural class. Focusing more on the nature of the functions in the problems, the characterisation of all tractable languages of cost functions was a long-standing problem which has recently been solved [32, 33], resolving positively the so-called Feder and Vardi Conjecture [34] that there is a P/NP-complete dichotomy for the constraint satisfaction problem parameterized by the language of possible constraint relations [35]. In these polynomial languages, submodular CFNs stand out. A submodular CFN is entirely composed of submodular functions, a typical example of which is the "soft equality" function, a function that is satisfied only when its two arguments are identical (and otherwise takes a constant positive valuation). This function can encode the polynomial time

Min-Cut problem as a CFN, and strong soft arc consistencies such as VAC or OSAC solve this problem directly, without requiring branch-and-bound [23].

4 Connections to Other Paradigms

The Semiring and Valued constraint satisfaction perspective extend beyond constraint programming and connects to several other areas of artificial intelligence, computer science and mathematics.

In constraint programming, the Semiring and Valued constraint frameworks offer a natural extension from feasibility/satisfaction to optimization. This path has also been followed by the "Constraint Optimization Programming" (COP [36]). In this framework, optimization is tackled as a succession of feasibility problems using increasingly strengthened requirements on a dedicated "criteria" or cost variable, connected to the rest of the problem by suitable constraints, using the global `sum` constraint for additive problems. In this elegant and general approach, the operator $\otimes$ is integrated inside the constraint language itself, as a dedicated global constraint. However, soft arc consistencies typically derive much stronger bounds than those offered by domain or bound consistencies applied to a cost variable. The reason is that soft local consistencies exchange marginal cost information between variables whereas domain consistencies rely only on shrinking domains, which is less informative. With the increasing number of global constraints available in soft constraint programming framework such as `pytoulbar2` [37], the computational edge of soft constraint programming over COP is becoming increasingly strong, as shown for example on Quadratic Assignment Problems [38].

While most scientific results in physics and mathematics rely on algebraic structures such as groups, rings and fields, more exotic algebraic structures such as semi-rings, semi-groups (aka monoids) and ordered semi-rings (also called dioids) have been repeatedly used to generalize algorithms in mathematics and computer science, including shortest path problems in graphs, variable elimination and message passing algorithms in Graphical Models [19] and more [39]. In particular, (non-serial) dynamic programming algorithms [16] have been generalized using a full semiring structure (without the additional requirements of c-semirings on $\oplus$) and shown to correctly solve optimization and counting problems on complex hypergraph structures [40, 41], implicitly covering non-serial dynamic programming for soft constraint and discrete Graphical Models optimization [19] as sub-cases. Semirings are also central in the representation and manipulation on uncertainty and incompleteness of information in triangular norms and t-conorms [42].

In the context of Graphical Models, the exact but worst-case exponential time and space non-serial dynamic programming algorithms have been simplified to so-called *message passing* algorithms, which have also been generalized to semirings [43]. Constraint networks being a specific subcase of discrete Graphical Models, the SCSP Soft Arc Consistency algorithm [14] is actually closely related to message passing in Graphical Models: algebraic message passing algorithms are correct when $\otimes$ is idempotent or for acyclic problems, but non convergent otherwise. They are nevertheless used even in non-convergent settings leading to heuristic practically extremely useful algorithms [44].

This connection between message passing in Graphical Models and (soft) arc consistency can be pushed further. Soft arc consistency algorithm for Valued networks [21, 24–28] are closely related to so-called convergent "Message Passing" (MP) algorithms in Graphical Models [45, 46] for the $(\mathbb{R}, +, \max)$ c-semiring. Similar convergent MP algorithms had actually been introduced decades sooner, based on linear programming and block-coordinate-descent, in the context of image analysis and so-called 2D-grammars, in Russian [47]. These were made accessible in English only decades later [30]. The Equivalent Preserving Transformations used in Soft Arc Consistency were also quickly adopted in the weighted MaxSAT world, leading to the definition of Max-resolution [48–50]. Finally, Virtual Arc Consistency can also be interpreted as a generalization to non-Boolean domains of the "roof-dual lower bound" [51] introduced for Quadratic Pseudo Boolean Optimization (with soft constraints involving at most two variables) and later generalized roof-dual bounds [52] (for larger arities).

These many connections highlight the repeated role of semirings, dioids and monoids [39] as common abstractions for combining and propagating information, bridging constraint reasoning, probabilistic inference, and optimization. Such a unifying view is increasingly relevant in the context of modern AI, where hybrid approaches combining symbolic reasoning and statistical learning are gaining importance.

5 Applications and Impact

The Semiring and Valued frameworks have been applied to a wide range of domains, including resource allocation, scheduling, network optimization, and decision support. Their flexibility in modeling preferences and trade-offs makes it particularly suitable for multi-criteria decision making, where different objectives must be balanced, a situation that benefit from the ability of c-semiring to deal with partially ordered structure for which dedicated Valued constraint network algorithms have also been recently proposed [53].

Specifically, the Semiring approach has been used to model trust, risk, and security levels in distributed systems [54], as well as quality of service in network routing [55], providing a natural way to reason about quantitative aspects of system behavior. More concrete applications of the SCSP framework include quality of service (QoS) routing in networks, where multiple criteria such as delay, bandwidth, and reliability must be optimized simultaneously [55]. In this context, c-semiring structures naturally capture the combination and comparison of different metrics, enabling flexible and expressive routing policies.

Another important application domain is the analysis of security protocols. SCSP soft constraints have been used to model different levels of security properties, such as confidentiality and authentication, allowing for a graded notion of security rather than a purely Boolean one [56]. This approach provides a more nuanced analysis of protocol behavior, capturing varying degrees of vulnerability and trust.

Valued constraint networks processing algorithms have been used to solve resource allocation problems [3], and more specifically applied to problems in computational biology, including genetics [57], polyploid genome assembly [58], and protein design [59, 60]. This last domain of application led to the creation of completely new proteins, including predicted viral variants [61], self-assembling symmetrical proteins [62–64] and SARS-CoV2 neutralizing binders [64].

6 Semiring and Valued Constraint Programming

Beyond their role as a modeling framework, SCSP soft constraints have also influenced the design of programming languages, leading to extensions of constraint logic programming and concurrent constraint programming. One significant development in this direction is Semiring Constraint Logic Programming (SCLP), where the classical notion of constraint satisfaction is replaced by soft constraints defined over an $\otimes$-idempotent c-semiring structure [65]. In this setting, the operational semantics of CLP is extended to account for the combination and comparison of valuations, allowing programs to reason not only about feasibility but also about optimality and preference.

Another important line of work is Soft Concurrent Constraint programming, which extends the concurrent constraint programming paradigm to handle soft constraints [66]. In classical concurrent constraint programming, computation proceeds through the interaction of agents that add (tell) and query (ask) constraints in a shared store. The soft extension generalizes this model by associating semiring-based valuations with constraints, thus enabling agents to reason under partial satisfaction, preferences, and uncertainty.

In Soft Concurrent Constraint programming, the constraint store evolves by combining information according to the c-semiring operations, and the notion of entailment is replaced by a comparison based on the c-semiring ordering. This allows for more flexible interaction patterns, where agents can operate even in the presence of inconsistent or suboptimal information, and progressively refine the quality of solutions. These language-level developments highlight the expressive power of the c-semiring framework: it is not only a tool for modeling optimization problems, but also a foundation for declarative programming paradigms that integrate optimization, concurrency, and interaction. More broadly, c-semiring-based languages contribute to bridging the gap between specification and computation, enabling high-level descriptions of problems that can be directly executed while preserving a clear semantic interpretation of preferences and trade-offs.

Further extensions of this line of work include the introduction of time into soft concurrent constraint programming, leading to frameworks such as Timed Soft Concurrent Constraint Programming [67]. These models allow the representation of temporal aspects together with preferences and uncertainty, enabling the specification of dynamic systems where both timing constraints and quality measures are relevant. More recently, c-semirings have been used for the design of concurrent languages to reason about complex systems, further extending the applicability of the framework to modern distributed and interactive scenarios.

Thanks to their ability to efficiently deal with non-idempotent $\otimes$ operator and more specifically the $(\mathbb{N} \cup \{\infty\}, +, \min)$ algebra, the Valued constraint network arc consistency algorithms have been integrated in Branch and bound algorithms, leading to the creation of the toulbar2 solver,[1] a state-of the art solver for the CFN/WCSP problem and all specific subcases of it. Toulbar2 is a C++ library which is used to provide a standalone solver that is able to read a variety of file formats. This includes computing Maximum a Posteriori realizations of stochastic Graphical Models such as Markov Random Fields (MAP/MRF) and Bayesian Networks (MPE/BN), Maximum Weighted Partial Satisfibility, or Quadratic Pseudo Boolean Optimization [68]. Toulbar2 constantly wins international competitions both in the Constraint Programming domain, for solving Constraint Optimization Problems and in the Graphical Model domain, for solving Maximum a Posteriori problems. To offer services closer to a Constraint Programming language, the toulbar2 C++ library is also accessible through Python bindings in the `pytoulbar2` library.

7 Future Directions

Several directions for future research emerge from this line of work. One important avenue is the integration of soft constraint reasoning with data-driven machine learning. The ability to represent preferences, uncertainty, and costs in a unified framework makes soft constraints a promising candidate for hybrid AI systems. Probabilistic reasoning is even more central to deep learning, which makes the non-idempotent $\otimes$ operators $\times$ and $+$ more specifically important (see also [69]). Such combinations have already been explored to extract criteria and preferences from sets of historical solutions [70], also using dedicated neuro-symbolic architectures combining deep learning with exact probabilistic reasoning [71, 72].

Another direction concerns explainability and interaction, where connections with argumentation frameworks may provide new ways to interpret and justify decisions based on soft constraints. A particularly promising direction concerns the integration of semiring-based soft constraints with argumentation frameworks. In this setting, c-semiring valuations can be used to associate weights, preferences, or probabilities to arguments and attacks, leading to weighted and probabilistic extensions of abstract argumentation [73]. This line of work opens new possibilities for combining logical reasoning with quantitative evaluation. Closely related is the development of probabilistic and weighted argumentation frameworks, where uncertainty and preferences are treated in a unified way. These approaches suggest a natural continuation of the c-semiring-based perspective in areas concerned with explainability and decision support. Another emerging direction is the design of concurrent and interactive languages for argumentation [74], where agents exchange and evaluate arguments under soft constraints. Such frameworks build upon the foundations of soft concurrent constraint programming, extending them toward richer models of interaction and reasoning.

[1] See github.com/toulbar2

8 Conclusion

After almost thirty years have passed, immense progresses have been done on the theoretical understanding and the construction of an algorithmic library for soft constraint networks. Non-serial dynamic programming algorithms can solve SCSPs, a complete characterization of "fair" valuation structures as idempotent stacking of additive layers [22] has been obtained, and correct arc consistency definitions and algorithms that can process both idempotent and non-idempotent $\otimes$-based networks have been introduced, leading to extremely fast exact solvers for solving soft constraint networks or the Maximum a Posteriori problems on stochastic Graphical Models.

These thirty years have also given the time to realize how research communities evolve independently, often ignoring and rediscovering similar fundamental ideas and results. The fact that semiring operations (and not just c-semirings) are sufficient to guarantee the proper behavior of variable elimination had been realized in the community of stochastic graphical models as soon as 1991 [40, 41]. Conversely, the first proper soft arc consistency algorithm for non-idempotent Valued constraint networks [21] seemingly predates closely related block-coordinate-descent-based convergent message passing algorithms for stochastic Graphical Models [45, 46] as well as the later related introduction of resolution for weighted MaxSAT [49, 50]. The Graphical Model, CP and SAT communities have also ignored the seminal work of M. Schlesinger and colleagues [47] on convergent message passing. For good reasons, as these results were published in Russian, while Ukraine was in USSR. But these results remain poorly cited even after they were made accessible in English [30], proving the strength of "Stigler's Law of Eponymy" (no scientific discovery is named after its original discoverer). As we explored these areas, also with our Ukrainian and Czech colleagues, it became clear that the many differences in terminology for the same concepts, called equivalent transformation here, reparametrization there and equivalent preserving transformations here,... slowly build a complex translation wall that probably explains a good fraction of this lack of recognition. The fact that the CP community (solving Weighted CSPs) and the ML community (solving MAP on Graphical models) arrived at similar destination without referencing the 1970s Kiev school is a testament to how robust the math is — but also how fragmented the history of science remains. The modern pressure to publish and get visible obviously does not help here, as it's ideal to unknowingly look as the discoverer of a new idea while simultaneously saving time by doing a light bibliography search. Among the possible disruptive effects of Large Language Models, some may think that such mathematical cryptomorphisms or simple lacks of recognition may become rarer. This is yet unclear.

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Data availability. Not applicable.

Code availability. Toulbar2 code is available at https://github.com/toulbar2/toulbar2, under the MIT licence.

Material availability. Not applicable.

Declarations

Competing interest. Thomas Schiex serves as scientific advisor and shareholder of the protein design startup Amineo. These roles have not influenced the design, analysis, or interpretation of the work presented here. All authors declare that there are no other competing interests.

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