Numbers and logic together in CP: A Practical View of Cost Function Networks

CP'2020 tutorial

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This is not a virtual tutorial

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ANITI
ARTIFICIAL & NATURAL INTELLIGENCE
TOULOUSE INSTITUTE

INRAE
science for people, life & earth

What is a (discrete) graphical model?

Informally (see also [CGS20])

A description of a multivariate function as the combination of a set of simple functions

WHAT FOR?

Concisely describing and analyzing complex systems

- Concise: we use a set of *small* functions
- Complex: the joint function results from the interaction of several small functions

Example

- A digital circuit — value of the output
- A Sudoku grid — solution or not
- A schedule or a time-table — feasibility, acceptability
- A pedigree with genotypes [SGS08] — Mendel consistency, probability
- A frequency assignment [Cab+99] — interference amount
- A 3D molecule [All+14] — energy, stability

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Notations

- Variables: $X, Y, Z, \dots$, possibly indexed as X_i
- Domains: D^X for variable X, or D^i for variable X_i
- Unknown values: $u, v, w, x, y, z \dots$
- Sequence of variables: $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \dots$
- Sequence of possible values: $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \dots$
- Domain of a sequence of variables $\boldsymbol{X} : D^{\boldsymbol{X}}$ (Cartesian product of the domains)
- $\boldsymbol{u} \in D^{\boldsymbol{X}}$ is an assignment of $\boldsymbol{X}$ (a value for each variable in $\boldsymbol{X}$)
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Definition (Graphical Model (GM [Bis06; KF09]))

A GM $\mathcal{M} = \langle \boldsymbol{V}, \Phi \rangle$ is defined by:

- a sequence of variables $\boldsymbol{V}$ — n
- each $X \in \boldsymbol{V}$ has finite domain D^X — max size d
- a set Φ of functions (or factors) — e
- Each function $\varphi_{\boldsymbol{S}} \in \Phi$ is a function from $D^{\boldsymbol{S}} \to B$ — scope $\boldsymbol{S}$, arity $|\boldsymbol{S}|$

Definition ($\mathcal{M}$ joint function)

$$\Phi_{\mathcal{M}}(\boldsymbol{v}) = \bigoplus_{\varphi_{\boldsymbol{S}} \in \Phi} \varphi_{\boldsymbol{S}}(\boldsymbol{v}[\boldsymbol{S}])$$

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How are functions $\varphi_S \in \Phi$ represented?

- Default: as tensors over B (multidimensional tables)
- Boolean vars: (B-weighted) clauses (disjunction of variables or their negation)
- Arithmetic, polynomes [BH02]
- Predicates (ALL-DIFFERENT [Rég94; LL12],...)

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Constraint networks [RBW06]/SAT [BHM09] $B = \mathbb{B} = \{t, f\}, \oplus = \wedge$

- a sequence of domain variables V
- a set Φ of e Boolean functions (or constraints)
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$\mathcal{M}$ defines a joint Boolean feasibility/consistency function

$$\Phi_{\mathcal{M}} = \bigwedge_{\varphi_S \in \Phi} \varphi_S$$

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Markov Random Fields: $B = \mathbb{R}^+, \oplus = \times$

- a set V of domain variables
- a set Φ of potential functions
- $\varphi_S \in \Phi : \prod_{X \in S} D^X \to \mathbb{R}^+$

$\mathcal{M}$: induces a probability distribution

$$\Phi_{\mathcal{M}} = \prod_{\varphi_S \in \Phi} \varphi_S \qquad P_{\mathcal{M}} \propto \Phi_{\mathcal{M}}$$

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Cost Function Networks $\mathcal{M}$ [FW92; SFV95; CS04]

$B = \overline{\mathbb{N}}^k, \oplus = \overset{k}{+}$

- a sequence of domain variables $\boldsymbol{V}$
- a set Φ of e numerical functions
- Each function $\varphi_{\boldsymbol{S}} \in \Phi$ is a function from $D^{\boldsymbol{S}} \to \overline{\mathbb{N}}^k$

- $\overline{\mathbb{N}}^k$: elements of $\mathbb{N} \cup \{\infty\}$ bounded by k — k finite or not
- $\overset{k}{+}$ is the k-bounded addition — $\alpha \overset{k}{+} \beta = \min(\alpha + \beta, k)$

$\mathcal{M}$ defines a joint (bounded) integer function

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CFN "normal form" — Used inside the solver

- Have a constant function $\varphi_\varnothing$
- Have all their unary functions $\varphi_i, X_i \in \boldsymbol{V}$ — $\varphi_i(u) = k$ means u deleted
- All functions have different scopes

Main properties

- $\varphi_\varnothing$ is a lower bound of the joint function $\Phi_{\mathcal{M}}$
- $k = 1$: Constraint networks and SAT, $\overset{k}{+}$ is $\wedge$

Graph $G = (\boldsymbol{V}, \boldsymbol{E})$ with edge weight function w

- A Boolean variable X_i per vertex $i \in V$
- A cost function per edge $e = (i,j) \in E : \varphi_{ij} = w(i,j) \times \mathbb{1}[x_i \neq x_j]$
- Hard edges: constraints with costs 0 or ∞ (when $x_i \neq x_j$)

A simple graph

- vertices $\{1, 2, 3, 4\}$
- cut weight 1
- edge $(1, 2)$ hard

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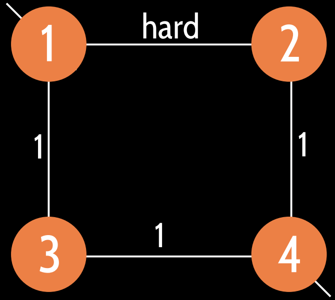

Example: Min-CUT with hard edges

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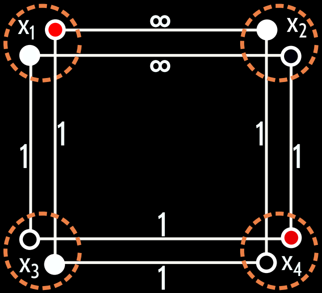

Min-CUT on 4 variables with hard edge

```
{
  problem :{name: "MinCut", mustbe: "<100.0"},
    variables: {x1: ["l"], x2: ["l","r"], x3: ["l","r"], x4: ["r"]}
    functions: {
      cut12: {scope: ["x1","x2"], costs: [0.0, 100.0, 100.0, 0.0]},
      cut13: {scope: ["x1","x3"], costs: [0.0,1.0,1.0,0.0]},
      cut23: {scope: ["x2","x3"], costs: [0.0,1.0,1.0,0.0]},
      cut34: {scope: ["x3","x4"], costs: [0.0,1.0,1.0,0.0]}
}
```

Definition (Functions and graphical models equivalence)

Two functions (or GMs) are equivalent iff they are always equal

Definition (Relaxation of a function or graphical model)

A function (or GM) φ is a relaxation of φ' iff $\varphi \leq \varphi'$

For $B = \mathbb{B}$, $t < f$

(φ relaxation of φ') $\Leftrightarrow$ ($\varphi' \models \varphi$)

Equivalence, relaxation

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Optimization

Minimization queries

- $B = \{t \equiv 0, f \equiv 1\}, \oplus = +^{1} = \wedge$, clauses — the SAT Problem
- $B = \{t \equiv 0, f \equiv 1\}, \oplus = +^{1} = \wedge$, tensors — the Constraint Satisfaction Problem
- $B = \overline{\mathbb{N}}^{k}, \oplus = +^{k}$, tensors — the Weighted Constraint Satisfaction Problem

We always use $+^{k}$

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The "local polytope" [Sch76; Kos99; Wer07] (without eq. (1))

$$\text{Minimize} \sum_{i,a} \varphi_i(a) \cdot x_{ia} + \sum_{\substack{\varphi_{ij} \in \Phi \\ a \in D^i, b \in D^j}} \varphi_{ij}(a,b) \cdot y_{iajb} \quad \text{such that}$$

$$\sum_{a \in D^i} x_{ia} = 1 \qquad \forall i \in \{1, \ldots, n\}$$

$$\sum_{b \in D^j} y_{iajb} = x_{ia} \qquad \forall \varphi_{ij} \in \Phi, \forall a \in D^i$$

$$\sum_{a \in D^i} y_{iajb} = x_{jb} \qquad \forall \varphi_{ij} \in \Phi, \forall b \in D^j$$

$$x_{ia} \in \{0, 1\} \qquad \forall i \in \{1, \ldots, n\} \qquad (1)$$

$nd + ed^2$ variables, $n + 2ed$ constraints

Conditioning: $\varphi_{\boldsymbol{S}|X=a} \quad (X \in \boldsymbol{S})$ **Assignment**

$\varphi_{\boldsymbol{S}|X=a}(\boldsymbol{v}) = (\varphi_{\boldsymbol{S}}(\boldsymbol{v} \cup \{X = a\})$ Scope $\boldsymbol{S} - \{X\}$, negligible complexity

		X_1		
	a	1	2	3
X_2	b	3	1	2
	c	2	3	1

Conditioning by $X_2 = b$

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Systematic tree search — Time $O(d^n)$, linear space

- If all $|D^X| = 1$ obvious minimum — update k to $\Phi_{\mathcal{M}}(\boldsymbol{v})$
- Else choose $X \in \boldsymbol{V}$ s.t. $|D^X| > 1$ and $u \in D^X$ and reduce to
  1. one query where we condition by $X_i = u$
  2. one where u is removed from D^X
- Return the minimum

Optimization — Branch and Bound [LW66]

If the $\underbrace{\text{local lower bound}}_{\varphi_\varnothing}$ reaches the $\underbrace{\text{global upper bound}}_{k}$ Prune!

Partial search

Relaxed pruning ($(1+\alpha)\varphi_\varnothing \geq k$) [Poh70], bounded number of backtracks or discrepencies (LDS [HG95])

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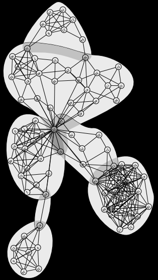
Partial search

Relaxed pruning ($(1+\alpha)\varphi_\varnothing \geq k$) [Poh70], bounded number of backtracks or discrepencies (LDS [HG95])

Hybrid Best First Search [All+15] Anyspace

- Uses Depth-First Search for a bounded amount of backtracks
- Pending nodes are pushed onto a list of Open nodes
- The next DFS starts from the best Open node
- Tree-decomposition friendly (BTD [GSV06]/AND-OR search [MD09])



Nice properties

- Good upper bounds quickly (DFS)
- A constantly improving lower bound (optimality gap)
- Implicit restarts, easy parallelization

Depth First (CP) or Best First (ILP)?

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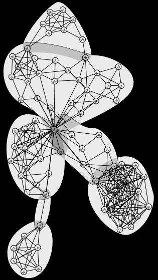

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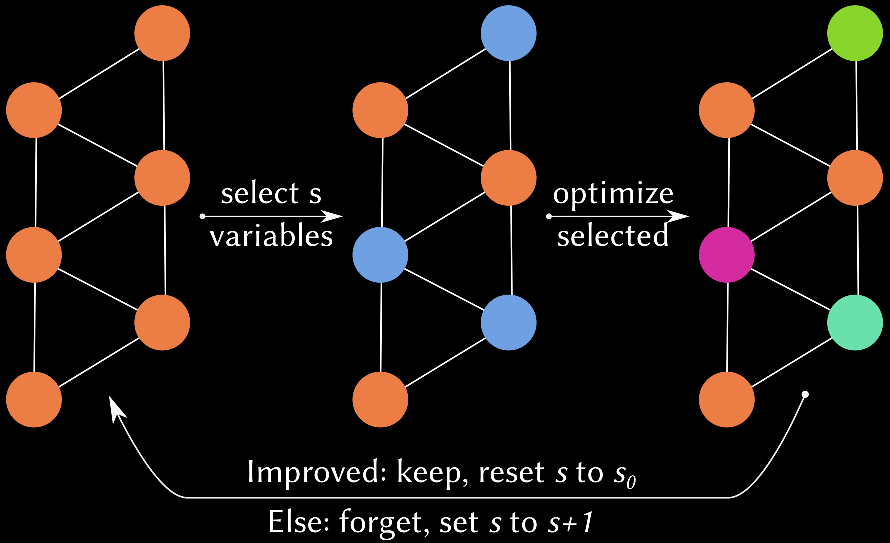
Also local search of course (VNS here)



Two last tools: Combination and Elimination

Combination of $\varphi_{\boldsymbol{S}}$ and $\varphi_{\boldsymbol{S'}}$ — Space/time $O(d^{|\boldsymbol{S}\cup\boldsymbol{S'}|})$ for tensors

$$(\varphi_{\boldsymbol{S}} \overset{k}{+} \varphi_{\boldsymbol{S'}})(\boldsymbol{v}) = \varphi_{\boldsymbol{S}}(\boldsymbol{v}[\boldsymbol{S}]) \overset{k}{+} \varphi_{\boldsymbol{S'}}(\boldsymbol{v}[\boldsymbol{S'}])$$

φ_{X_2}:

X_2	
a	4
b	6
c	4

$\varphi_{X_1 X_2}$:

X_2 \ X_1			
a	1	2	3
b	3	1	2
c	2	3	1

$\Longrightarrow$

X_2 \ X_1			
a	5	6	7
b	9	7	8
c	6	7	5

Elimination of $X \in \boldsymbol{S}$ from $\varphi_{\boldsymbol{S}}$ — Time $O(d^{|\boldsymbol{S}|})$, space $O(d^{|\boldsymbol{S}|-1})$ for tensors

$$\varphi_{\boldsymbol{S}}[-X](\boldsymbol{u}) = \min_{\boldsymbol{v}\in D^X} \varphi_{\boldsymbol{S}}(\boldsymbol{u} \cup \boldsymbol{v})$$

Produces relaxations

X_2 \ X_1			
a	5	6	7
b	9	7	8
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Eliminate X_2

X_1		
5	6	5

Eliminate X_1

$\varnothing$

5

Combination of φ_S and $\varphi_{S'}$ — Space/time $O(d^{|S \cup S'|})$ for tensors

$$(\varphi_S \overset{k}{+} \varphi_{S'})(v) = \varphi_S(v[S]) \overset{k}{+} \varphi_{S'}(v[S'])$$

X_2	φ
a	4
b	6
c	4

X_2 \ X_1	1	2	3
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$\Longrightarrow$

X_2 \ X_1			
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Elimination of $X \in S$ from φ_S — Time $O(d^{|S|})$, space $O(d^{|S|-1})$ for tensors

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Used together

- Combination accumulates all information in a single function
- Elimination forgets one variable without loosing *optimality* information

At the core of

- Local consistencies, Unit propagation: subproblem induced by one function
- Variable elimination, the Resolution Principle: subproblem around one variable

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ANITI INRAE

Arc consistency of X_i w.r.t. φ_{ij} [RBW06]

- Combine φ_{ij} and the unary φ_j
- Eliminate X_j producing a function (message) on X_i

$$m_i^j = (\varphi_{ij} \overset{k}{+} \varphi_j)[-X_j]$$

Properties

- The message can be added to φ_i (relaxation, value deletion)
- X_i is AC w.r.t. φ_{ij} if $m_j^i \leq \varphi_i$ (no new information)
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Messages can not be included in the CFN: loss of equivalence, meaningless result

Equivalence Preserving Transformations with $-^k$ $(\alpha -^k \beta) \equiv ((\alpha = k)\ ?\ k : \alpha - \beta)$

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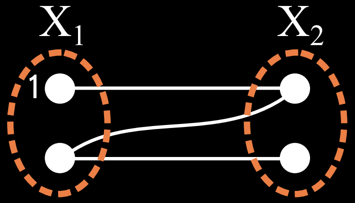
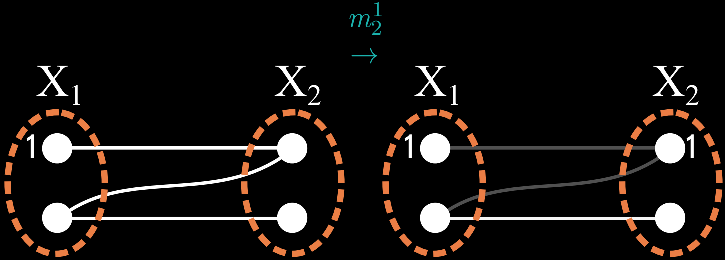


(Loss of) properties

Preserves equivalence but fixpoints may be non unique (or may not exist)



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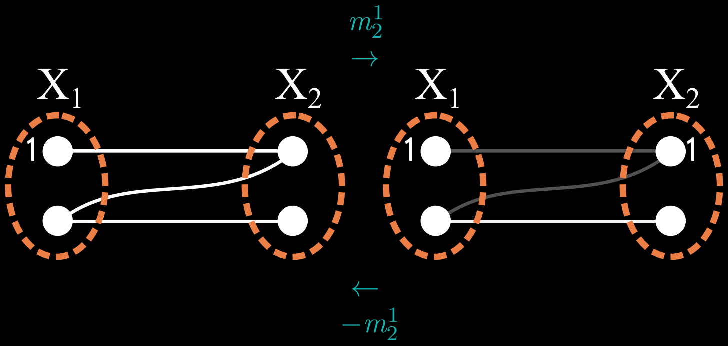

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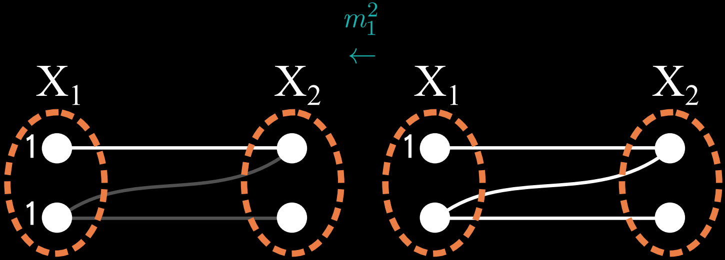

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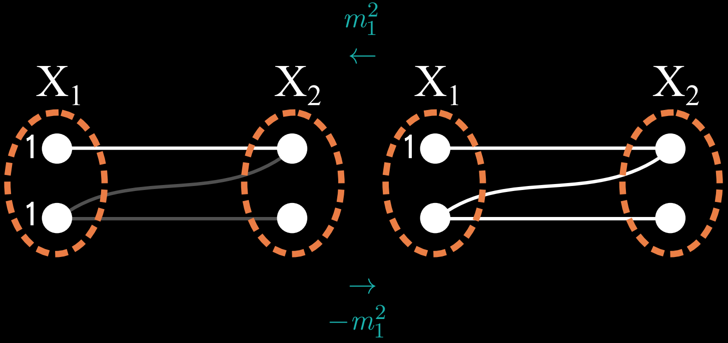

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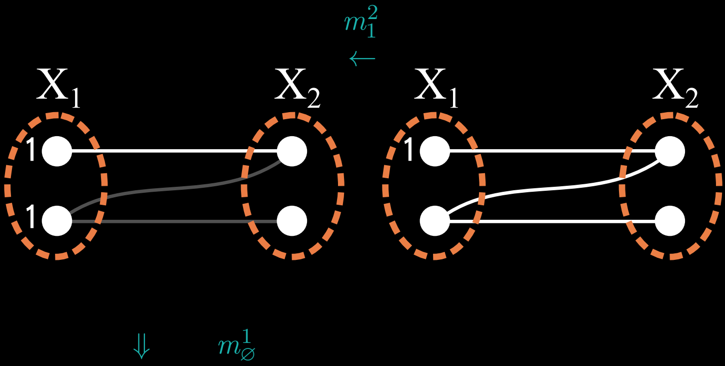

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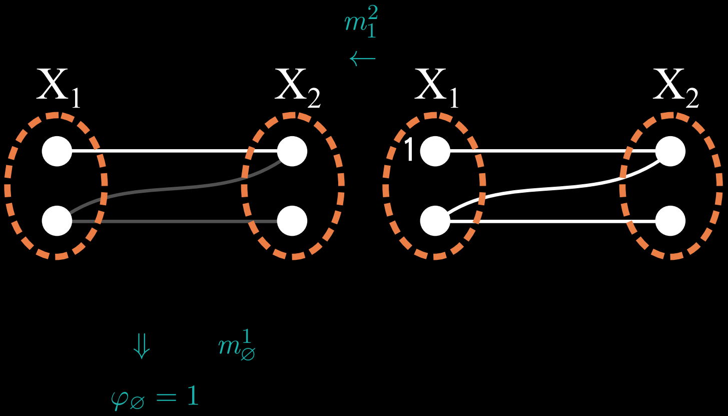

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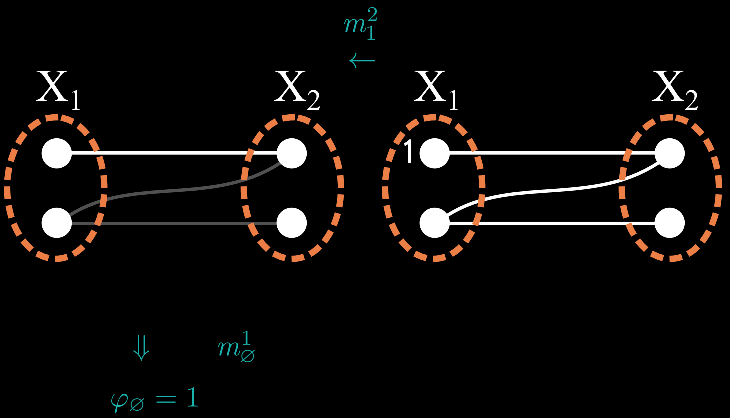

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Many way to avoid loops (enforce fixpoint existence)

ANITI INRAE

The many "soft ACs" [Coo+10]

- NC: one unary function [Lar02] — Unary supports ($\varphi_i(u) = 0$)
- +AC: one binary function [Sch00; Lar02] — Arc supports ($v \in D^j, \varphi_{ij}(u, v) = 0$)
- +DAC: FDAC, binary & unary function (+ direction) [Coo03] — Full Supports
- +Existential AC: EDAC, a star (variable incident functions) [Lar+05] — EAC supports
- +Virtual AC: any spanning tree [Coo+08; Coo+10] — VAC supports

Properties — Related works in Comp. Vision [Kol06; Son+12; Wer07; Kol15]

- Proper extension of classical NC/DAC or AC respectively — ($k = 1$)
- Polynomial time and $O(ed)$ space — (Generalized ACs)
- Incremental, strengthens $\varphi_\varnothing$ — (VAC $\geq$ EDAC $\geq$ FDAC $\geq$ AC $\geq$ NC)
- May have several fixpoints/$\varphi_\varnothing$

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Computing a sequence of integer EPTs that maximizes $\varphi_\varnothing$ is decision NP-complete [CS04]

Set of rational EPTs (OSAC [Sch76; Coo07; Wer07; Coo+10])

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Variables for a binary CFN, no constraints [Sch76; Kos99; CGS07; Wer07; Coo+10]

1. u_i: amount of cost shifted from φ_i to $\varphi_\varnothing$
2. p_{ija}: amount of cost shifted from φ_{ij} to $\varphi_i(a)$
3. p_{jib}: amount of cost shifted from φ_{ij} to $\varphi_j(b)$

OSAC

Maximize $\sum_{i=1}^{n} u_i$ subject to

$$\varphi_i(a) - u_i + \sum_{(\varphi_{ij} \in C)} p_{ija} \geq 0 \qquad \forall i \in \{1, \ldots, n\}, \forall a \in D^i$$

$$\varphi_{ij}(a,b) - p_{ija} - p_{jib} \geq 0 \qquad \forall \varphi_{ij} \in C, \forall (a,b) \in D^{ij}$$

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The "local polytope"

$$\text{Minimize} \sum_{i,a} \varphi_i(a) \cdot x_{ia} + \sum_{\substack{\varphi_{ij} \in \Phi \\ a \in D^i, b \in D^j}} \varphi_{ij}(a,b) \cdot y_{iajb} \quad \text{such that}$$

$$\sum_{a \in D^i} x_{ia} = 1 \qquad \forall i \in \{1, \ldots, n\} \qquad (2)$$

$$\sum_{b \in D^j} y_{iajb} = x_{ia} \qquad \forall \varphi_{ij} \in \Phi, \forall a \in D^i \qquad (3)$$

$$\sum_{a \in D^i} y_{iajb} = x_{jb} \qquad \forall \varphi_{ij} \in \Phi, \forall b \in D^j \qquad (4)$$

u_i multiplier for (2), p_{ija}/p_{jib} for (3) and (4)

Local polytope proved to be "Universal for LP" [PW15]

Problem solved by OSAC/VAC [Coo+10; KZ17]

- Tree-structured problems
- Permutated submodular problems (eg. Min-Cut, Min/Max-closed relations)
- OSAC/VAC + $\forall X_i, \exists! u \in D^i$ s.t. $\varphi_i(u) = 0$ [Coo+10; HSS18; TGK20]

Supports provide value ordering heuristics

- EAC supports u for X_i: $\varphi_i(u) = 0$, can be extended for free on X_i's star
- VAC supports can be extended for free on any spanning tree [Kol06; Coo+08; Coo+10]

NC provides cost-based pruning

If $(\varphi_\varnothing \overset{k}{+} \varphi_i(u)) = k$, NC deletes u

Local consistencies vs. LP

- OSAC empirically very expensive to enforce
- Local consistencies provide fast approximate LP bounds
- and deal with constraints seamlessly

CFN Local Consistencies

Enhance CP with fast incremental approximate Linear Programming dual bounds

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CPLEX V12.4.0.0

```
Problem '3e4h.LP' read.
Root relaxation solution time =  811.28 sec.
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tb2 and VAC (AC3 based)

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Kind words from OpenGM2 developpers

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ANITI INRAE

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What if the language is CNF?

Soft UP and Max resolution [LH05; BLM07] — More issues

- combination and elimination are Ok
- but subtracting a clause from another clause does not yield a clause (CNF/DNF)
- generates additional "compensation" clauses [LH05; HLO07; BLM07; LHG08])

ANITI INRAE

Definition (Message from X to its neighbors)

Let $X \in \boldsymbol{V}$, and Φ^X be the set $\{\varphi_{\boldsymbol{S}} \in \Phi \text{ s.t. } X \in \boldsymbol{S}\}$, $\boldsymbol{T}$, the neighbors of X.

The message $m_{\boldsymbol{T}}^{\Phi^X}$ from Φ^X to $\boldsymbol{T}$ is:

$$m_{\boldsymbol{T}}^{\Phi^X} = (\sum_{\varphi_{\boldsymbol{S}} \in \Phi^X}^{k} \varphi_{\boldsymbol{S}})[-X]$$

The message contains all the effect of X on the optimization problem Distributivity

$$\min_{\boldsymbol{v} \in D^{\boldsymbol{V}}} \left[\sum_{\varphi_{\boldsymbol{S}} \in \Phi}^{k} (\varphi_{\boldsymbol{S}}(\boldsymbol{v}[\boldsymbol{S}])) \right] = \min_{\boldsymbol{v} \in D^{\boldsymbol{V} - \{X\}}} \left[\sum_{\varphi_{<\boldsymbol{S}} \in \Phi - \Phi^X \cup \{m_{\boldsymbol{T}}^{\Phi^X}\}}^{k} (\varphi_{\boldsymbol{S}}(\boldsymbol{v}[\boldsymbol{S}])) \right]$$

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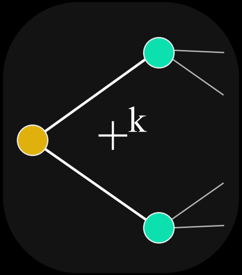
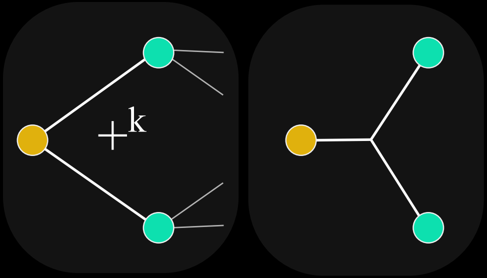


Daoopt & mini-buckets [DR03] split Φ^X in subsets of controlled size (lower bound)



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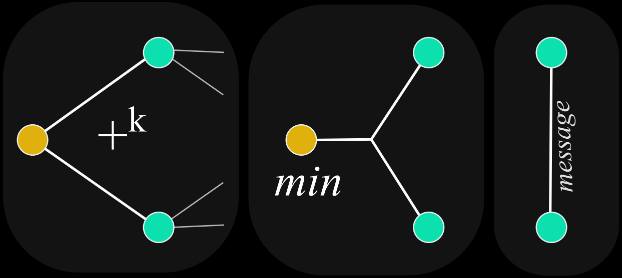

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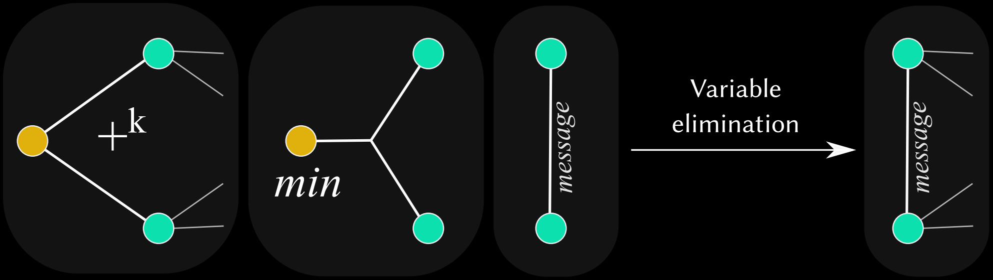

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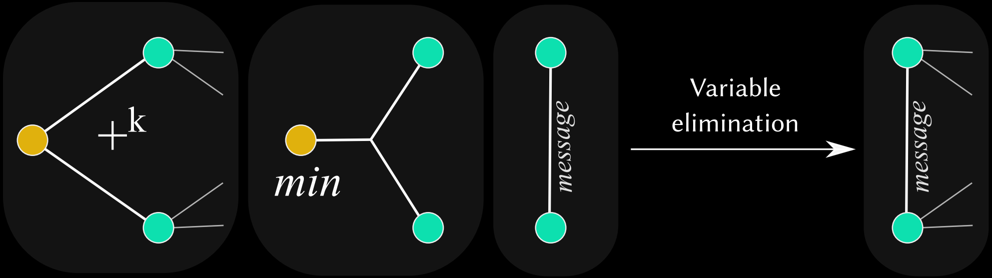

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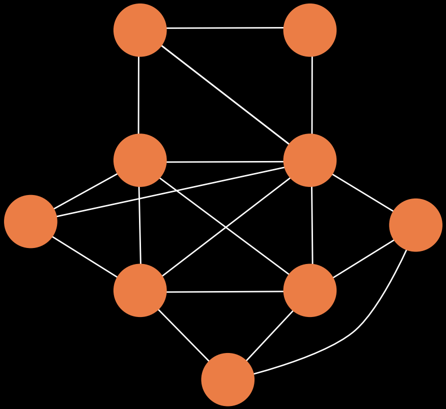
Boosting search with VE [Lar00]

If a variable has a small degree, eliminate it (backtrackable) else branch



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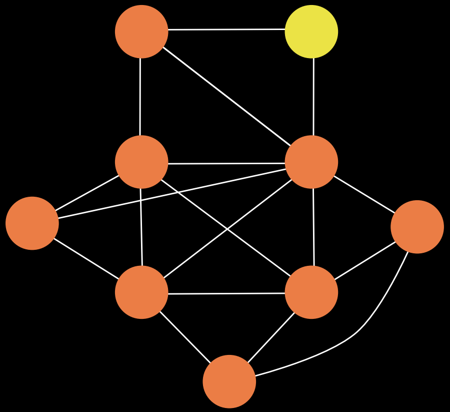

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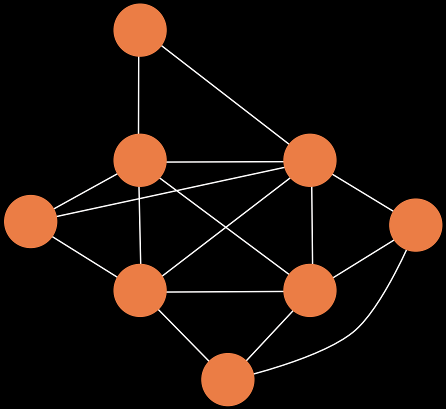

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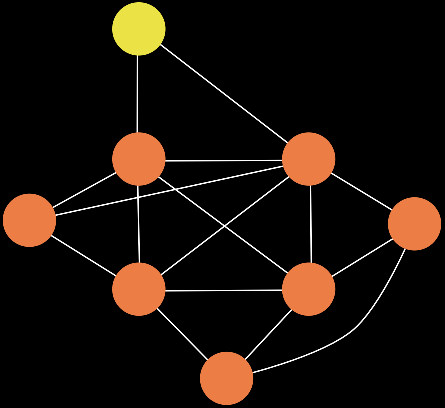

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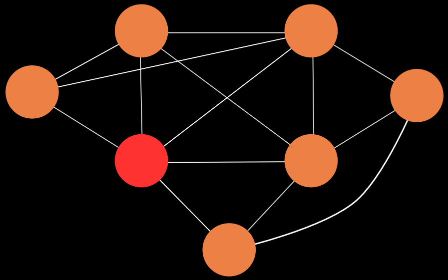
If a variable has a small degree, eliminate it (backtrackable) else branch



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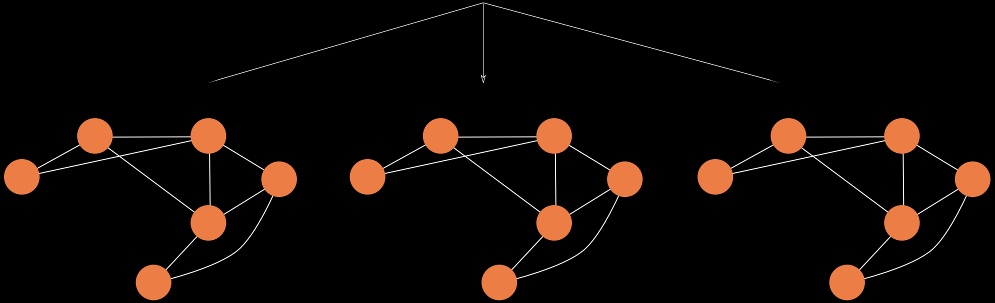

Presentation Outline

Additional algorithmic ingredients

- Variable ordering: weighted degree [Bou+04], last conflict [Lec+09], VAC-based [TGK20]
- Value ordering (for free): existential or virtual supports
- Dominance analysis (substitutability/DEE) [Fre91; DPO13; All+14]
- Function decomposition [Fav+11]
- Global cost functions (weighted Regular, All-Diff, Among...) [LL12; All+16]
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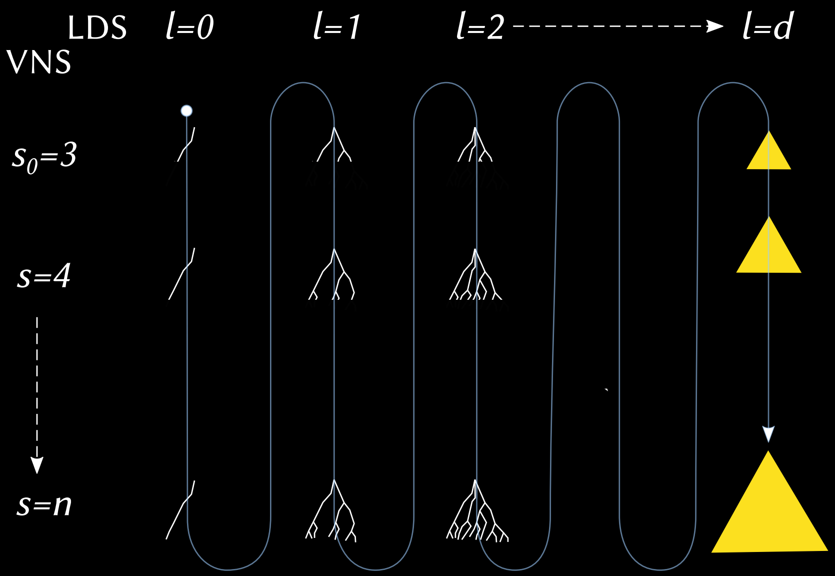
Unified Decomposition Guided VNS [Oua+20; Oua+17]



Practical aspects

- C++ Open source, MIT licence on GitHub, available in Debian
- Uses 64 bits integer costs to represent adjustable precision decimal costs
- Tackles minimization, maximization with costs of arbitrary signs and constraints
- JSON compatible CFN input format
- Python API (PyToulbar2)

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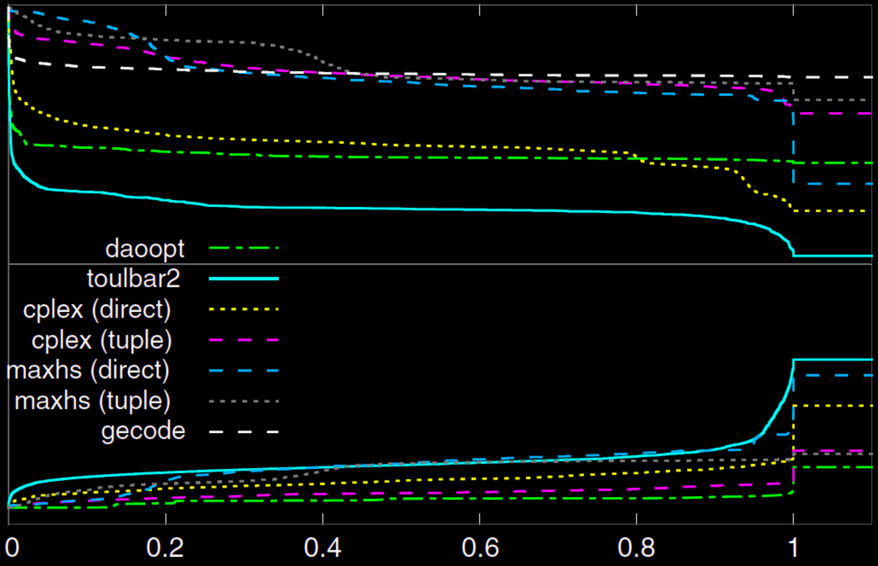
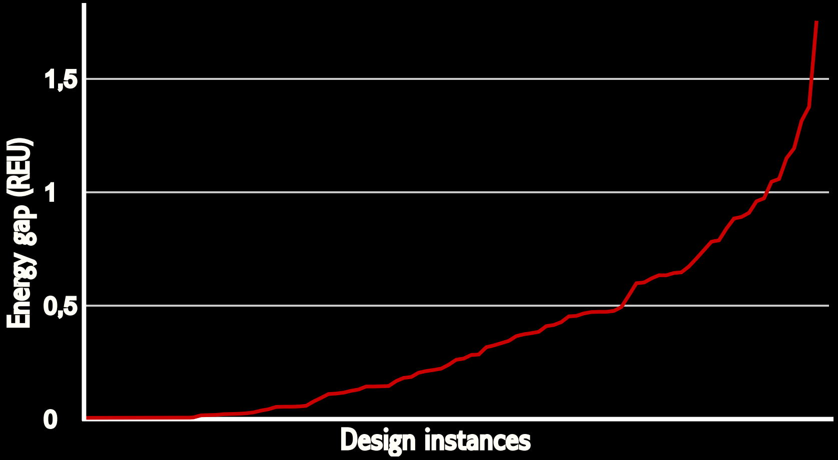
3026 instances of various origins genoweb.toulouse.inra.fr/~degivry/evalgm

- MRF: Probabilistic Inference Challenge 2011
- CVPR: Computer Vision & Pattern Recognition OpenGM2
- CFN: Cost Function Library
- MaxCSP: MaxCSP 2008 competition
- WPMS: Weighted Partial MaxSAT evaluation 2013
- CP: MiniZinc challenge 2012/13

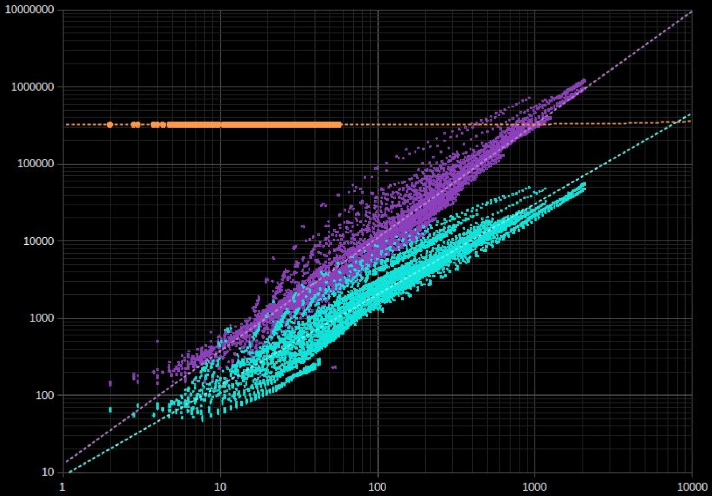
Benchmark	Nb.	UAI	WCSP	LP(direct)	LP(tuple)	WCNF(direct)	WCNF(tuple)	MINIZINC
MRF	319	187MB	475MB	2.4G	2.0GB	518MB	2.9GB	473MB
CVPR	1461	430MB	557MB	9.8GB	11GB	3.0GB	15GB	N/A
CFN	281	43MB	122MB	300MB	3.5GB	389MB	5.7GB	69MB
MaxCSP	503	13MB	24MB	311MB	660MB	73MB	999MB	29MB
WPMS	427	N/A	387MB	433MB	N/A	717MB	N/A	631MB
CP	35	7.5MB	597MB	499MB	1.2GB	378MB	1.9GB	21KB
Total	**3026**	0.68G	2.2G	14G	18G	5G	27G	1.2G


daoopt
toulbar2
cplex (direct)
cplex (tuple)
maxhs (direct)
maxhs (tuple)
gecode
0
0.2
0.4
0.6
0.8
1

Comparison with Rosetta's Simulated annealing [Sim+15]



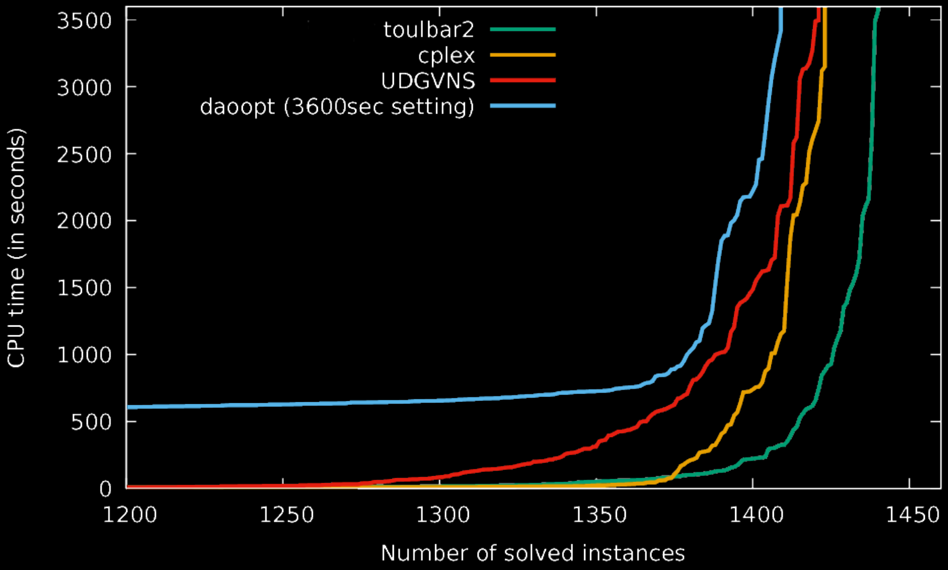
Optimality gap of the Simulated annealing solution as problems get harder

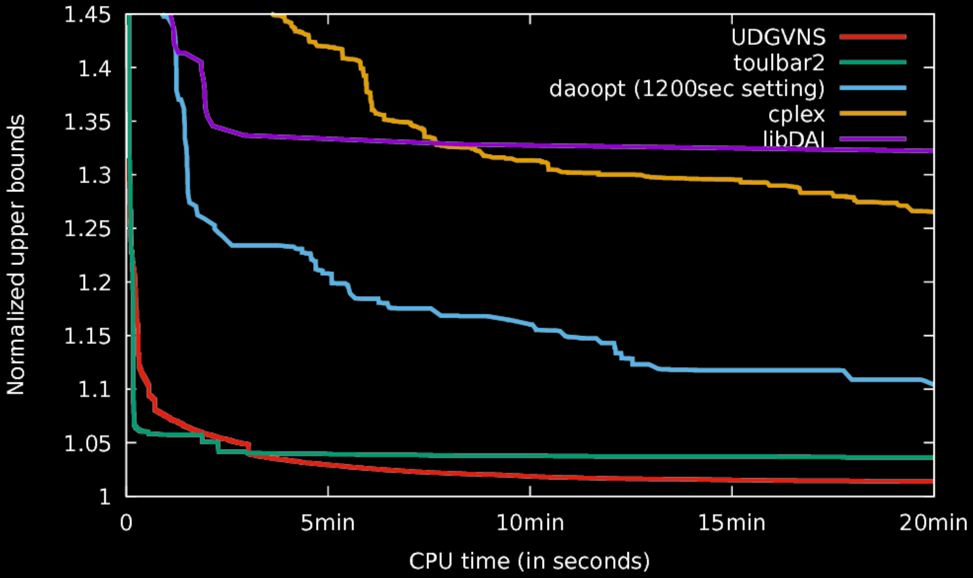


DWave approximations

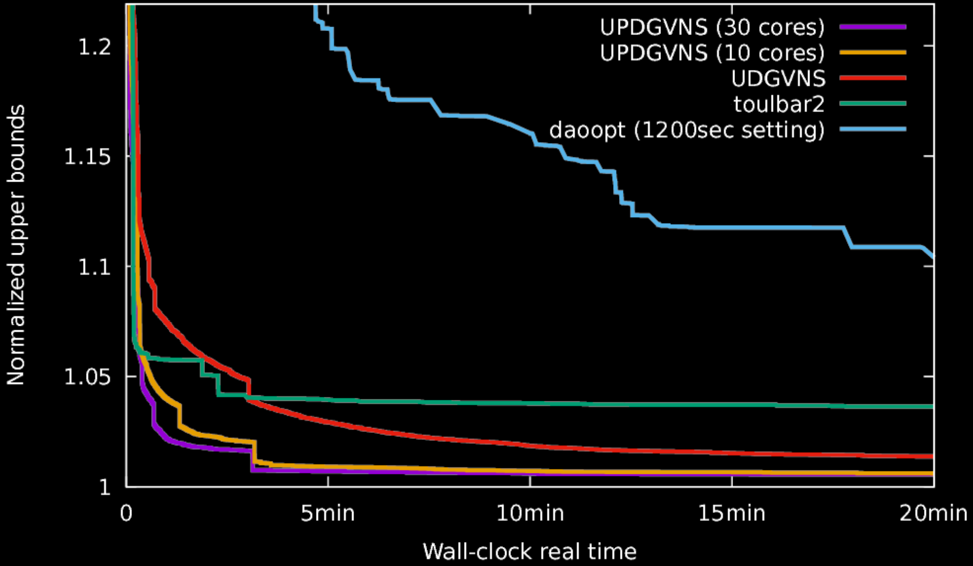
within 1.16 of optimum, 10% of the time | 4.35, 50% of the time | 8.45, 90% of the time

UDGVNS - Number of solved problems [Oua+17]




UDGVNS
toulbar2
daoopt (1200sec setting)
cplex
libDAI
Normalized upper bounds
1.45
1.4
1.35
1.3
1.25
1.2
1.15
1.1
1.05
1
0
5min
10min
15min
20min
CPU time (in seconds)

UPDGVNS - Upper bound profiles[Oua+20]



Definition (Learning a pairwise CFN from high quality solutions)

Given:

- a set of variables V,
- a set of assignments E i.i.d. from an unknown distribution of high-quality solutions

Find a pairwise CFN $\mathcal{M}$ that can be solved to produce high-quality solutions

MRFs tightly connected to CFNs ($k = \infty$) (additive energy)

$$\text{MRF } \mathcal{M} \xrightarrow[-\log(x)]{} \text{CFN } \mathcal{M}^{\ell} \xrightarrow[\exp(-x)]{} \text{MRF } \mathcal{M}$$

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Opens the door to learning from data $\boldsymbol{E}$

- E a set of i.i.d. assignments of $\boldsymbol{V}$
- The log-likelihood of $\mathcal{M}$ given $\boldsymbol{E}$ is $\log(\prod_{\boldsymbol{v}\in\boldsymbol{E}} P_{\mathcal{M}}(\boldsymbol{v})) = \sum_{\boldsymbol{v}\in\boldsymbol{E}} \log(P_{\mathcal{M}}(\boldsymbol{v}))$
- Maximimizing loglikelihood over all binary $\mathcal{M}$ $\quad (O(\frac{n(n-1)}{2} d^2)$ costs)

Maximum loglikelihood $\mathcal{M}$ on $\mathcal{M}_\ell$

$$\begin{aligned}\mathcal{L}(\mathcal{M},\boldsymbol{E}) &= \log(\textstyle\prod_{\boldsymbol{v}\in\boldsymbol{E}} P_{\mathcal{M}}(\boldsymbol{v})) = \sum_{\boldsymbol{v}\in\boldsymbol{E}} \log(P_{\mathcal{M}}(\boldsymbol{v}))\\ &= \textstyle\sum_{\boldsymbol{v}\in\boldsymbol{E}} \log(\Phi_{\mathcal{M}}(\boldsymbol{v})) - \log(Z_{\mathcal{M}})\\ &= \underbrace{\sum_{\boldsymbol{v}\in\boldsymbol{E}}(-C_{\mathcal{M}^\ell}(\boldsymbol{v}))}_{\text{-costs of } \boldsymbol{E} \text{ samples}} \underbrace{- \log(\sum_{t\in\prod_{X\in\boldsymbol{V}} D^X} \exp(-C_{\mathcal{M}^\ell}(\boldsymbol{t})))}_{\text{Soft-Min of all assignment costs}}\end{aligned}$$

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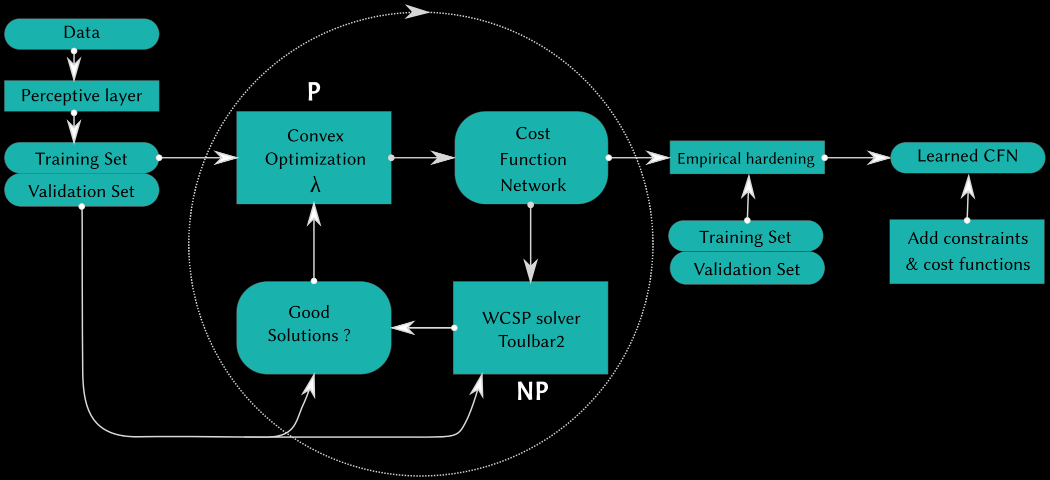
&= \textstyle\sum_{\boldsymbol{v}\in\boldsymbol{E}} \log(\Phi_{\mathcal{M}}(\boldsymbol{v})) - \log(Z_{\mathcal{M}}) \\
&= \underbrace{\sum_{\boldsymbol{v}\in\boldsymbol{E}} (-C_{\mathcal{M}^\ell}(\boldsymbol{v}))}_{\text{-costs of } \boldsymbol{E} \text{ samples}} \underbrace{- \log(\sum_{\boldsymbol{t}\in\prod_{X\in\boldsymbol{V}} D^X} \exp(-C_{\mathcal{M}^\ell}(\boldsymbol{t})))}_{\text{Soft-Min of all assignment costs}}
\end{aligned}$$

Learning a Cost Function Network from high-quality solutions



See how it learns how to play the Sudoku (and more) Friday 9/11, 1PM session

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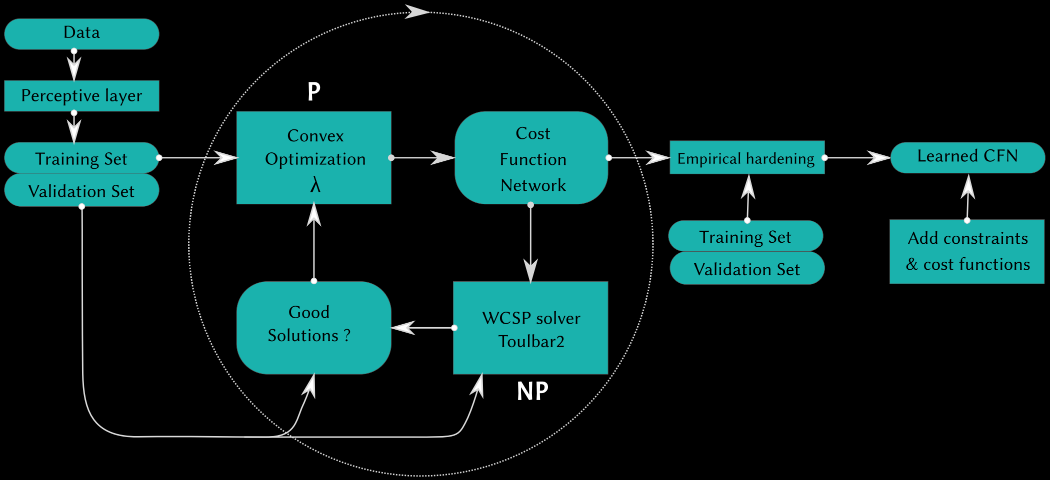

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