Cost Function Networks

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Martin Cooper
IRIT, Toulouse

Simon de Givry
INRA, Toulouse

Cost Function Networks

- What are they and why do we need them?
- Tractable classes
- Search using soft consistency
- Hybrid tree and local search
- Toulbar2 solver demonstration

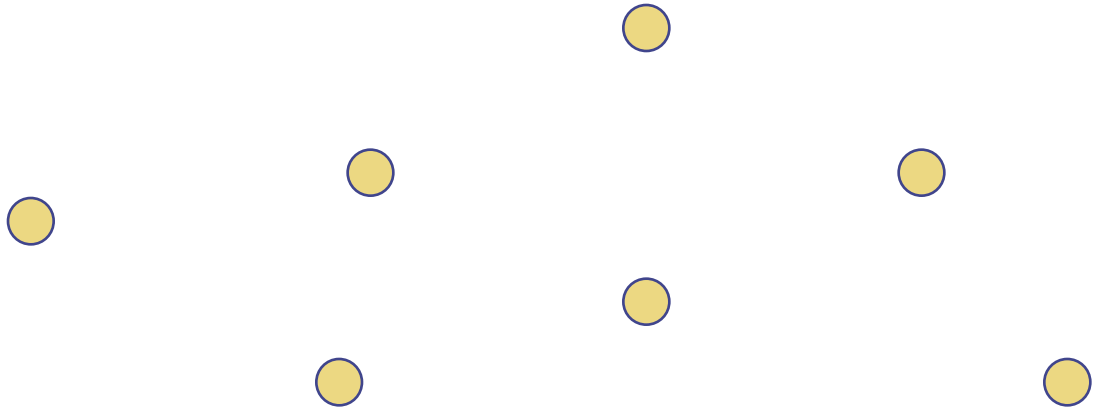
Chapter 1. What are CFNs?

Motivation,

Definitions,

Some general theorems

A unifying abstraction: CSP



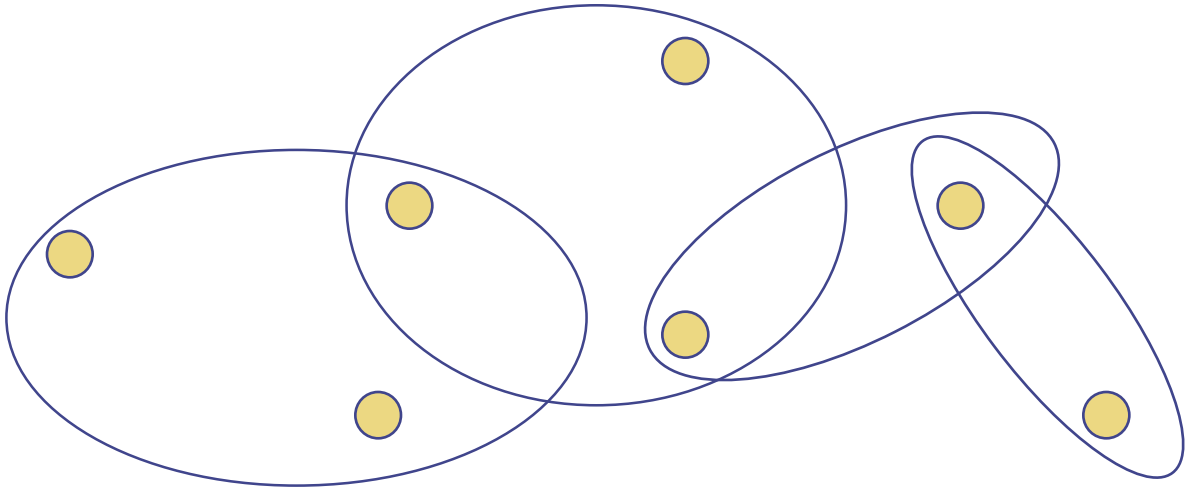
Variables = Talks to be scheduled at conference

Transmitters to be assigned frequencies

Amino acids to be located in space

Circuit components to be placed on a chip

A unifying abstraction: CSP



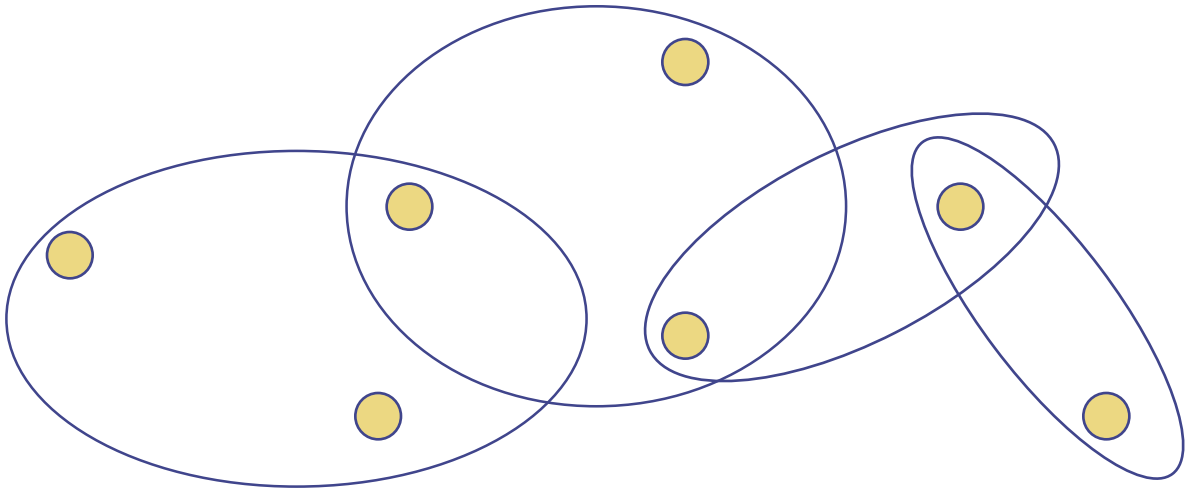
Constraints = All invited talks on different days

No interference between near transmitters

x + y + z > 0

Foundations dug before walls built

A unifying abstraction: CSP



A solution is an assignment of values to variables that satisfies all the constraints

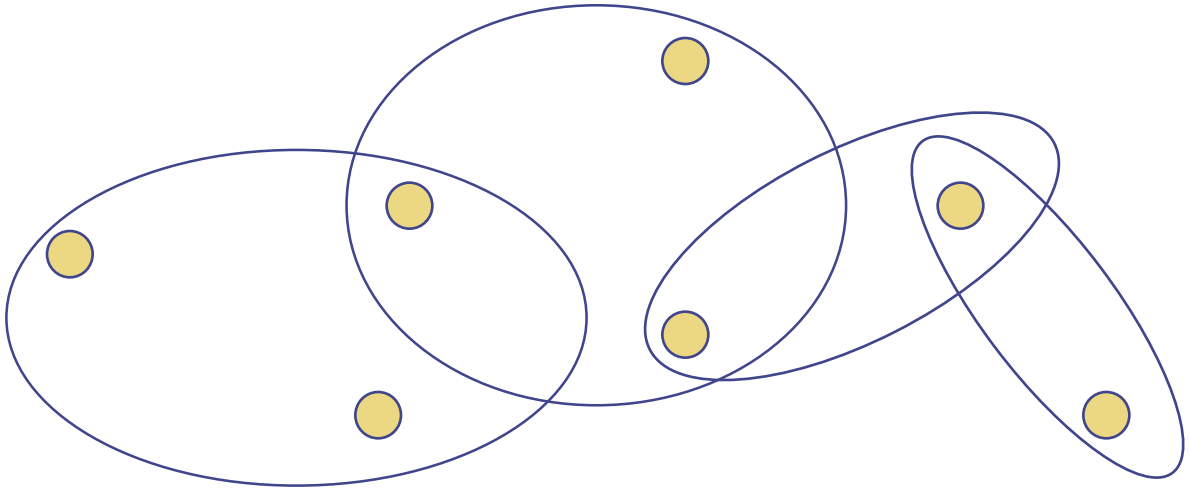
But what if…

- There are lots of solutions, but some are better than others?
- There are no solutions, but some assignments satisfy more constraints than others?
- We don’t know the exact constraints, only probabilities, or fuzzy membership functions?
- We’re willing to violate some constraints if we can get a better overall solution that way?

Fragmentation of research

- Constraint Optimisation Problem
- Max-CSP
- Max-SAT
- WCSP
- Fuzzy CSP
- Pseudo-Boolean Optimisation
- Bayesian Networks
- Markov Random Fields
- Integer Programming
- …

A unifying abstraction: CFN



Constraints ⬭ now associate costs with each assignment

A solution is an assignment of values to variables that minimises the combined costs

Definition of a CFN instance

- a set of n variables X_i with domains d_i
- a set of valued constraints, where each constraint has a
  - scope (list of variables)
  - cost function (function from assignments to costs)

It only remains to specify what the possible costs are, and how to combine them

Definition of a valuation structure

- a set S of costs
- a total order $<$
- minimum and maximum elements:
  we denote these by 0 and ∞
- an aggregation operator $\oplus$ which is commutative, associative, monotonic, and such that $\forall \alpha,\ \alpha \oplus 0 = \alpha$

Examples of valuation structures

- If S = {0, ∞}, then CFN ≡ CSP
- If S = non-negative reals ∪ {∞}, and ⊕ is addition, then CFN generalizes MAX-CSP.

  And we can model MPE in Bayesian networks or MAP in Markov random fields, by calculating the sum of -log(P(x | parents(x))
- If S = [0,1], and ⊕ is max, then CFN ≡ Fuzzy CSP
- If S = {0, 1, …., k}, and ⊕ is bounded addition $+_k$ where $\alpha +_k \beta = \min\{k, \alpha+\beta\}$, then CFN ≡ WCSP

Families of valuation structures

A valuation structure is idempotent if

$\forall \alpha,\ \alpha \oplus \alpha = \alpha$

All idempotent valuation structures are equivalent to Fuzzy CSP

Families of valuation structures

A valuation structure is strictly monotonic if

$\forall \alpha < \beta,\ \forall \gamma < \infty,\ \ \alpha \oplus \gamma < \beta \oplus \gamma$

A valuation structure is fair if

aggregation has a partial inverse, that is,

$\forall \alpha \geq \beta,\ \exists \gamma$ such that $\beta \oplus \gamma = \alpha$

All strictly monotonic valuation structures can be embedded in a fair valuation structure

Families of valuation structures

A valuation structure is discrete if between any pair of finite costs there are finitely many other costs

All discrete and fair valuation structures
can be decomposed into
a contiguous sequence of valuation structures
with aggregation operator $+_k$

Bibliography

- For general background on different formalisms for soft constraints, see the chapter on "*Soft Constraints*" by Meseguer, Rossi and Schiex, in the *Handbook of Constraint Programming*, Elsevier, 2006, or

  Cooper, de Givry & Schiex, "Réseaux de contraintes valuées", Chapter 7, *Panorama de l'Intelligence Artificielle*, Cépaduès, 2014.

- For classification results on valuation structures see "*Arc Consistency for Soft Constraints*", Cooper & Schiex, *AIJ*, 2004.

Chapter 2. Tractable classes

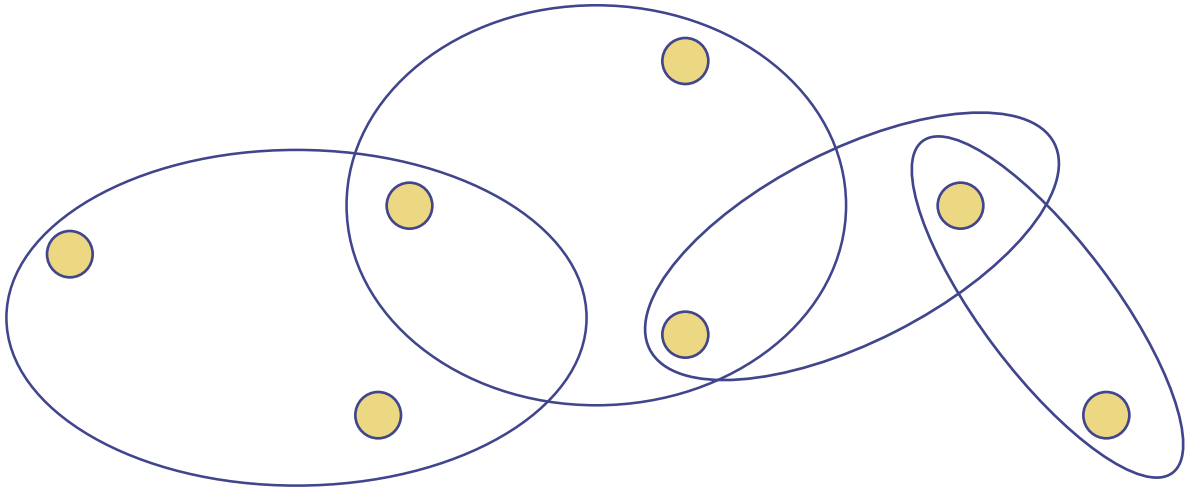
Structural restrictions,
Cost function languages,
Submodularity,
Weighted polymorphisms,
Hybrid tractability

General question

Having a unified formulation allows us to ask *general* questions about efficiency:

When is the CFN problem tractable?

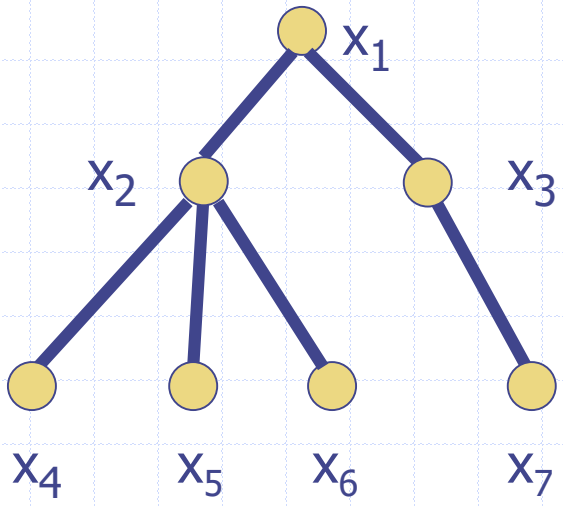
Problem features



- This picture illustrates the constraint scopes
- The set of scopes is sometimes called the constraint hypergraph, or the scheme
- Restricting the scheme can lead to tractability, as in the standard CSP

Structural tractability

Tree-structured binary CFNs are tractable



Time complexity $O(e\ d^2)$
Space complexity $O(n\ d)$

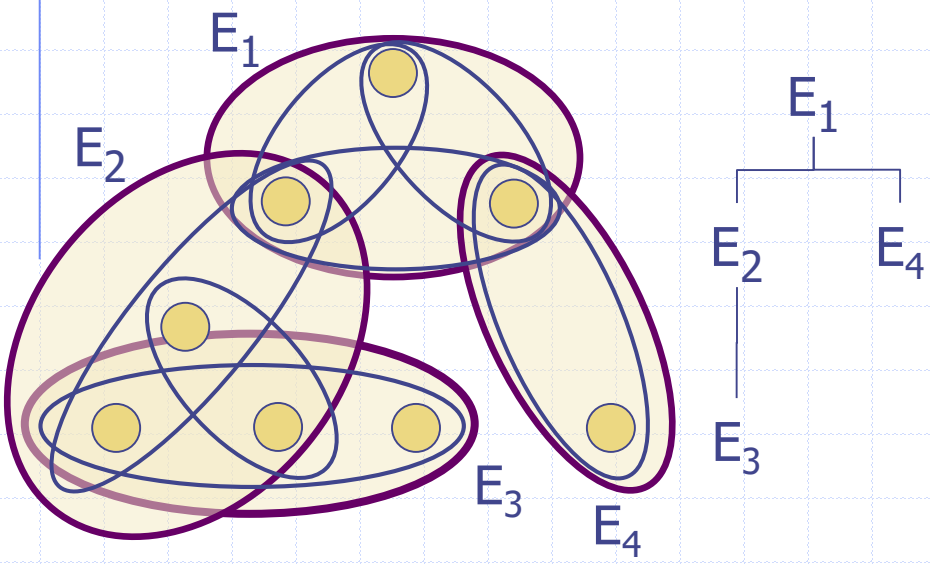
n: number of variables
d: maximum domain size
e: number of cost functions

Proceed from the leaf nodes to a chosen root node

Project out leaf nodes by minimising over possible assignments

Tree decomposition

Bounded treewidth CFNs are tractable



Time complexity $O(e\ d^{w+1})$
Space complexity $O(n\ d^{s})$

w: bounded treewidth
= max |Ei| - 1

s: max $\{|E_i \cap E_j|: i \neq j\}$

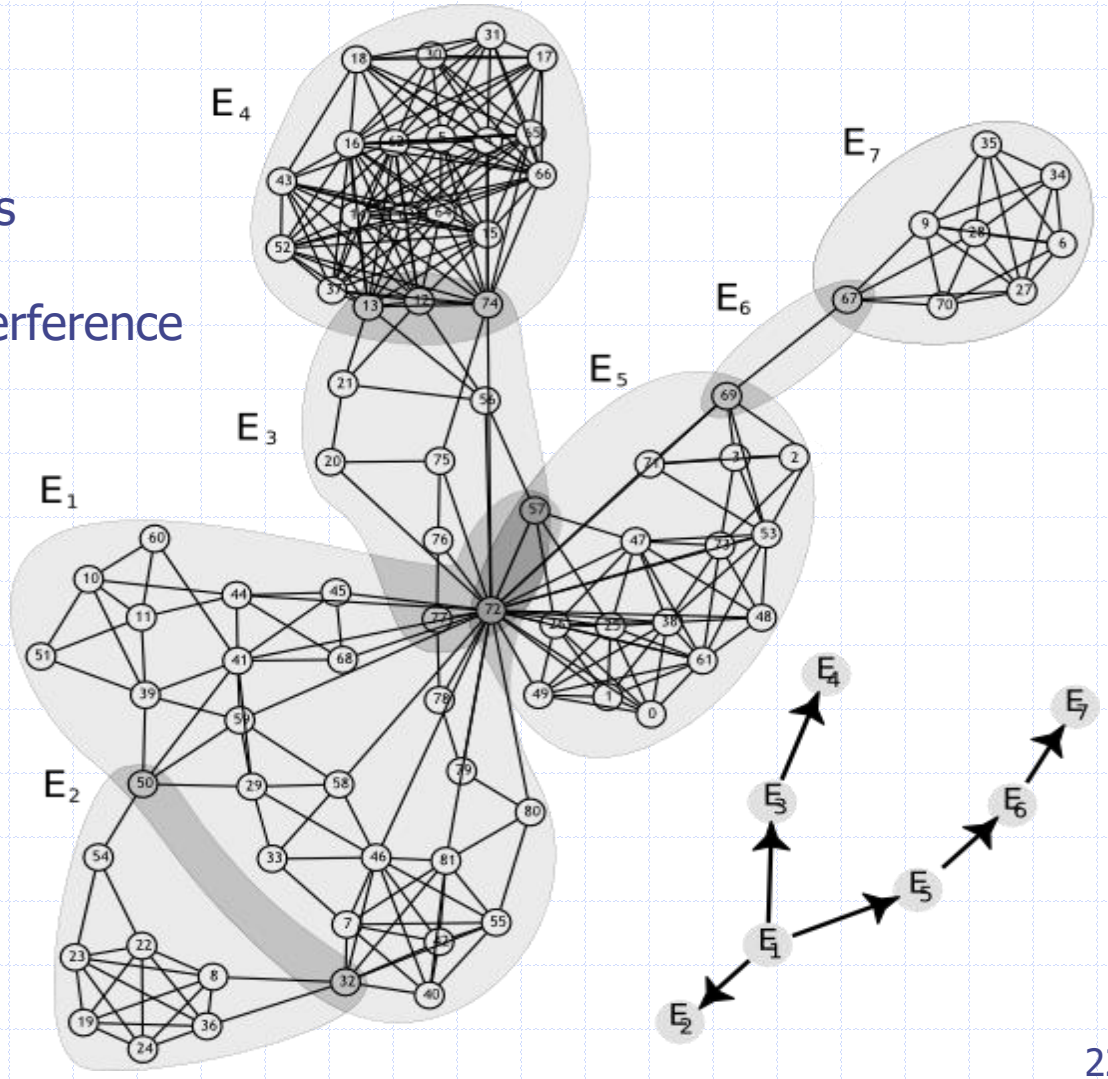
Finding a tree decomposition with minimum w* is NP-hard!

Tree decomposition example

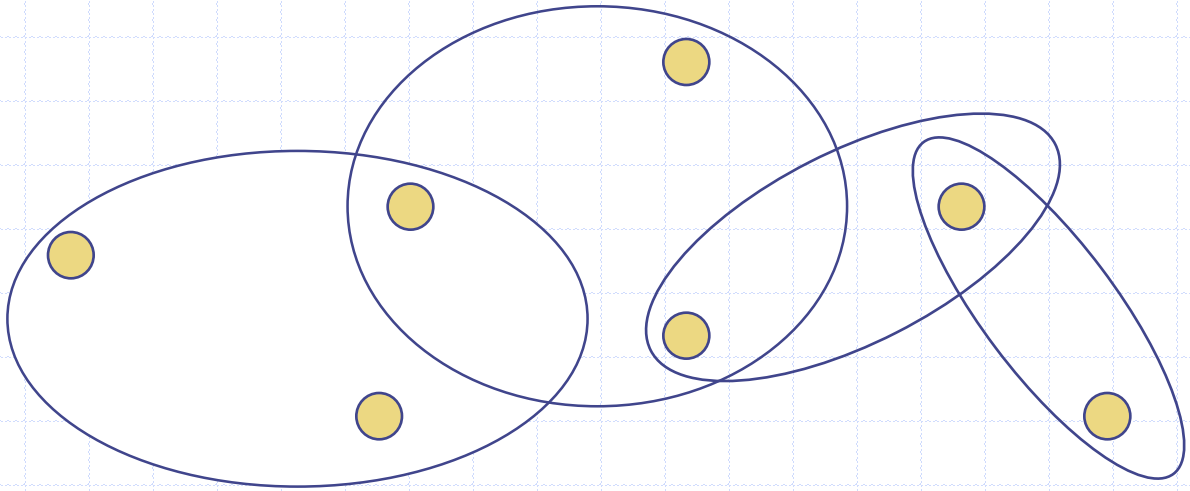
Benchmark problem
assigning frequencies
to transmitters
to minimise total interference

CELAR scen06r
n = 82
d = 44
e = 327

w = 26
s = 3

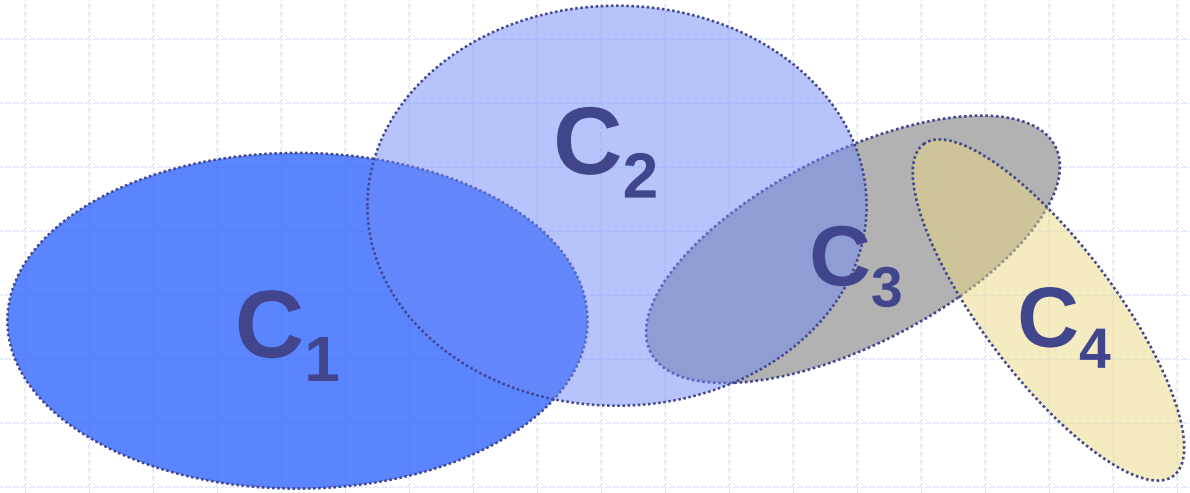


Problem features



- We have seen that structural features of a problem can lead to tractability
- This is very similar to the standard CSP
- What about other kinds of restrictions to the CFN?

More problem features



- The picture now emphasises the cost functions
- Restricting the cost functions we allow can also lead to tractability

Languages of cost functions

- A set of cost functions is called a language of cost functions
- CFN(Γ) represents the set of CFN instances whose cost functions belong to the language Γ
- For some choices of Γ, CFN(Γ) is tractable
- We consider examples where the valuation structure contains non-negative real values and infinity, and aggregation is standard addition

Submodular functions

A class of functions that has been widely studied in OR is the class of submodular functions…

A cost function c is submodular if $\forall \mathbf{s},\mathbf{t}$

$$c(\min(\mathbf{s},\mathbf{t})) + c(\max(\mathbf{s},\mathbf{t})) \leq c(\mathbf{s}) + c(\mathbf{t})$$

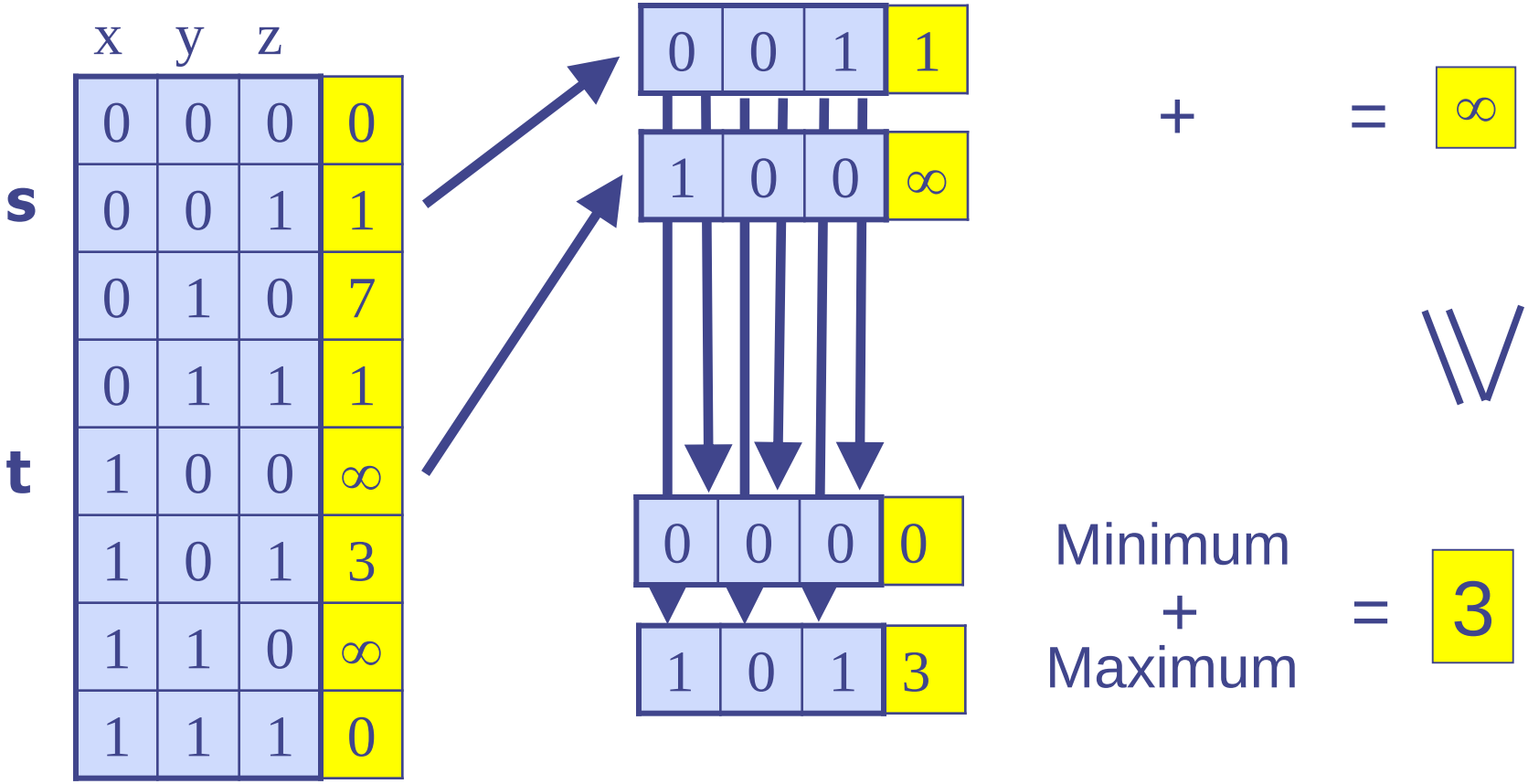
where min and max are applied component-wise, i.e.

$$\min(\langle s_1,\ldots,s_k\rangle,\langle t_1,\ldots,t_k\rangle) = \langle \min(s_1,t_1),\ldots,\min(s_k,t_k)\rangle$$

$CFN(\Gamma_{submodular})$ is tractable

Definition of submodularity

$\forall s,t \;\; c(\mathrm{Min}(s,t)) + c(\mathrm{Max}(s,t)) \leq c(s) + c(t)$



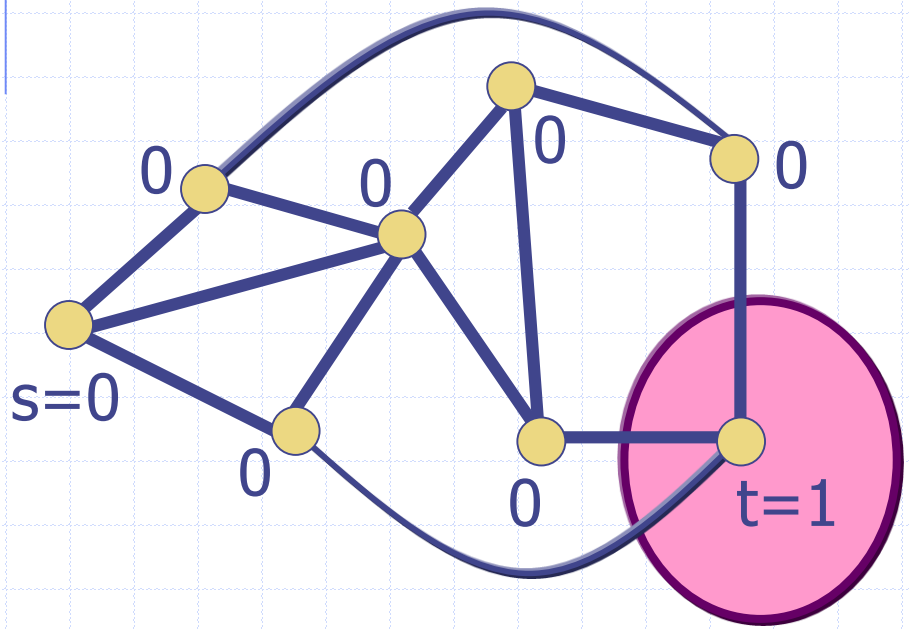
Examples of submodular functions

- all unary functions
- all linear functions (of any arity)
- the binary Boolean function ϕ_{cut}

  where $\phi_{cut}(a,b)=1$ if $(a,b)=(0,1)$ (0 otherwise)
- the rank function of a matroid
- the Euclidean distance function between two points (x_1, x_2), (x_3, x_4) in the plane
- $\phi(x,y)=(x-y)^r$ if $x \geq y$ (∞ otherwise) for $r \geq 1$

Example: Min-Cut

The s-t Min-Cut problem can be modelled using the single submodular binary cost function ϕ_{cut}



CFN with domain {0,1}

Valued constraints on all edges (both ways)

with cost function ϕ_{cut}

$\phi_{cut}(a,b)=1$ if $(a,b)=(0,1)$

Solution to CFN is a Min-Cut

Algorithms

- The best known *general* algorithm for Boolean submodular function minimisation is $O(e\, n^3 \log^{O(1)} n)$, where e=number of cost functions

  See Yin Tat Lee et al., "A Faster Cutting Plane Method and its Implications for Combinatorial and Convex Optimization", FOCS, 2015

- However, many special cases can be solved more efficiently by flow algorithms...

Boolean submodular functions

Many Boolean submodular functions can be expressed using the binary function ϕ_{cut}

(these include all {0,1}-valued Boolean submodular functions, all binary and all ternary Boolean submodular functions, and many others)

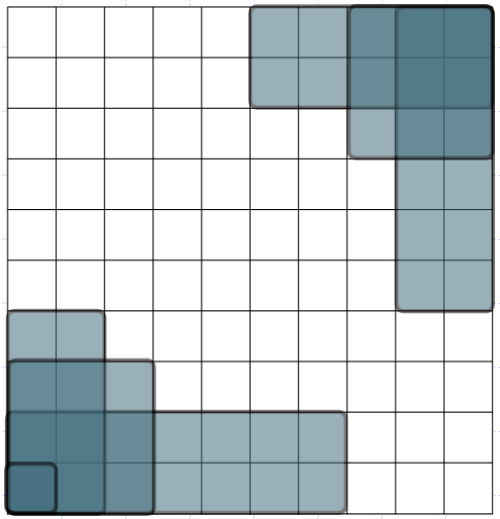
$CFN(\{\phi_{cut}\})$ is $O(n^3)$

See Živný & Jeavons, "Classes of submodular constraints expressible by graph cuts", *Constraints*, 2010

Binary submodular functions

Binary submodular functions
over any finite domain
can be expressed as a sum of
"Generalized Interval" functions

(they correspond to Monge matrices)



Binary CFN($\Gamma_{submodular}$) is $O(n^3d^3)$

See Cohen et al, "A maximal tractable class of soft constraints", *JAIR* 2004

Beyond submodularity

$\forall s,t \;\; c(\mathrm{Min}(s,t)) + c(\mathrm{Max}(s,t)) \leq c(s) + c(t)$

x	y	z	
0	0	0	0
0	0	1	1
0	1	0	7
0	1	1	1
1	0	0	∞
1	0	1	3
1	1	0	∞
1	1	1	0

We say that the cost function has the ***multimorphism*** (Min,Max)

By choosing *other* functions, we can obtain other tractable cost function languages…

Known tractable cases

If the cost functions all have one of these eight multimorphisms, then the problem is tractable:

1) (Min,Max)
2) (Max,Max)
3) (Min,Min)
4) (Majority,Majority,Majority)
5) (Minority,Minority,Minority)
6) (Majority,Majority,Minority)
7) (Constant 0)
8) (Constant 1)

See Cohen et al, "The complexity of soft constraint satisfaction", *AIJ* 2006

A dichotomy theorem

If the cost functions all have one of these eight multimorphisms, then the problem is tractable:

1) (Min,Max)
2) (Max,Max)
3) (Min,Min)
4) (Majority,Majority,Majority)
5) (Minority,Minority,Minority)
6) (Majority,Majority,Minority)
7) (Constant 0)
8) (Constant 1)

For Boolean cost functions...

In all other cases the cost functions have **no** significant common multimorphisms and the CFN problem is **NP-hard**.

See Cohen et al, "The complexity of soft constraint satisfaction", *AIJ* 2006

Benefits of a general approach

- This dichotomy theorem immediately implies earlier results for SAT, MAX-SAT, Weighted Min-Ones and Weighted Max-Ones
- Multimorphisms have also been used to show that not all submodular functions can be expressed using binary functions

  see Živný et al "The expressive power of binary submodular functions", *Discrete Applied Maths*, 2009.
- Multimorphisms (and its generalisations) allow submodularity to be generalised to larger classes of tractable languages.

Generalising multimorphisms

- Fractional/weighted polymorphism - each function has an associated rational weight >0 (in multimorphisms this weight is always 1)

  see Cohen et al, "An Algebraic Theory of Complexity for Discrete Optimization", SIAM J. Comput, 2013.

- This generalisation has led to ground-breaking results such as …

Recent theoretical results:

- Identification of all tractable languages of *finite-valued* cost functions.

  J. Thapper and S. Živný, J.ACM 2016.

- Necessary and sufficient condition for the tractability of any language (assuming the identification of all tractable crisp languages)

  V. Kolmogorov, A. Krokhin, M. Rolinek, SIAM J. Comput. 2017.

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- Identification of all tractable crisp languages
  A. Bulatov, FOCS 2017 and D. Zhuk, FOCS 2017.

Hybrid classes

- Certain important properties of instances cannot be defined by purely structural or purely language restrictions.

- *For example:* absence of saddle points implies a unique local minimum (but this is neither a structural nor a language property).

Truly hybrid tractability

- Truly hybrid = independent restrictions on the language Γ and on the constraint graph.

- *For example:*

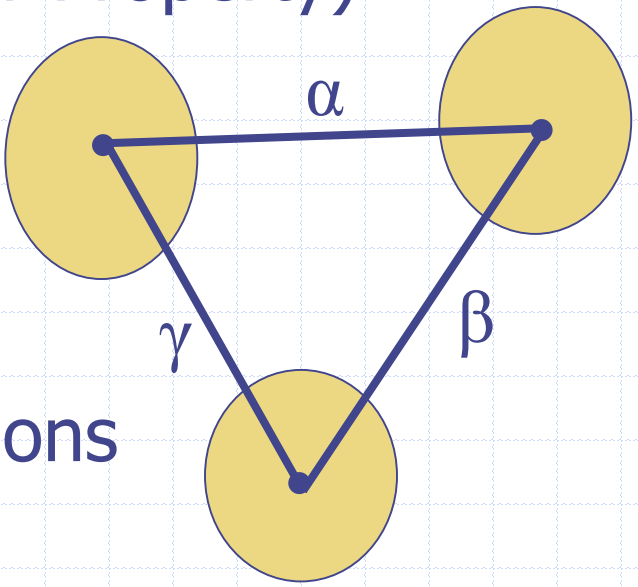
  Γ + planar $\Rightarrow$ no new tractable languages

  P. Fulla & S. Živný, MFCS 2016

Tractability from local patterns

- Example: JWP (Joint Winner Property)

$$\alpha \geq \min(\beta,\gamma)$$



+ arbitrary unary cost functions

- Generalisation:

  cross-free convex (CFC) ≈ convex cardinality cost functions on nested scopes

Bibliography

- For general background on tractable structures, see the chapter on "Tractable Structures" by R. Dechter, in the *Handbook of Constraint Programming*, Elsevier, 2006.

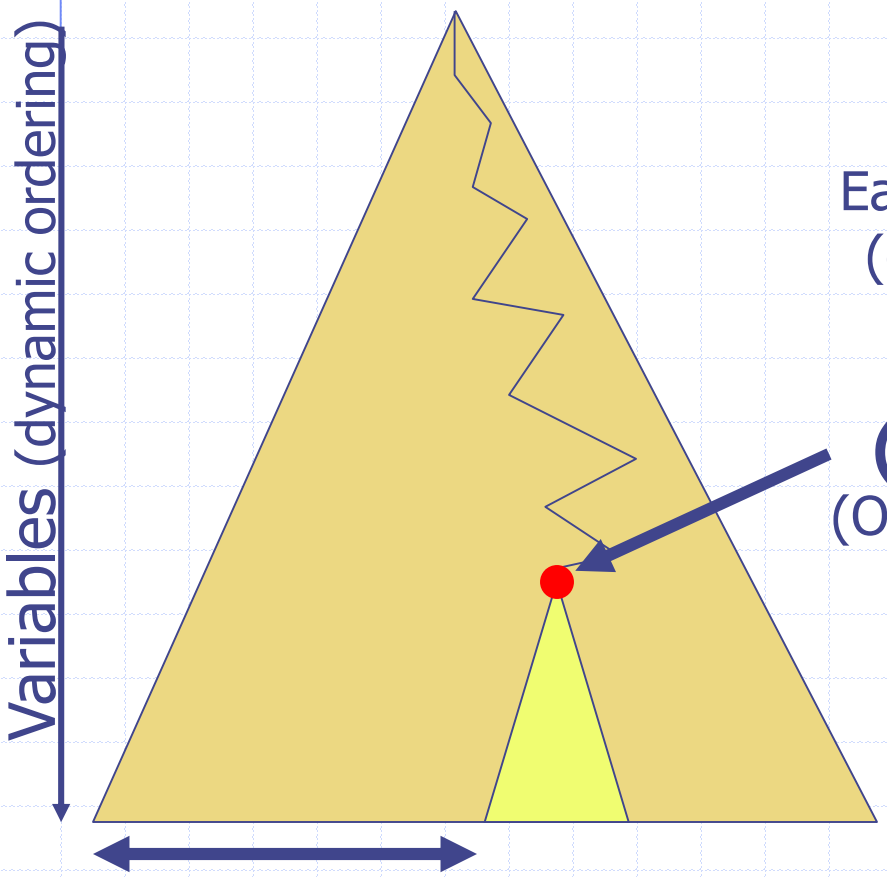
- For language and hybrid tractability see

  A. Krokhin & S. Živný, "The Constraint Satisfaction Problem: Complexity and Approximability". *Dagstuhl Follow-Ups*, 2017.

Chapter 3. Search using problem transformations

Branch and Bound,
Equivalence-preserving operations,
Soft local consistency (node, arc, existential, virtual, optimal),
Soft global constraints.

DepthFirst Branch and Bound (DFBB)



Each node is a CFN subproblem
(on the unassigned variables)

(LB) Lower Bound = c_0
(Obtained by enforcing local consistency)
= underestimate of the best solution in the sub-tree

If $c_0 \geq UB$ then prune

(UB) Upper Bound
= best solution found so far

Equivalence-preserving transformations (EPT)

- An EPT transforms CFN instance P_1 into another CFN instance P_2
- with the same objective function.
- Examples of EPTs:

  - Propagation of inconsistencies (∞ costs)
  - UnaryProject
  - Project/Extend

UnaryProject(i,α)

Precondition: $0 \leq \alpha \leq \min\{c_i(a) : a \in d_i\}$

$c_0 := c_0 + \alpha$;
for all $a \in d_i$ **do**
    $c_i(a) := c_i(a) - \alpha$;

Increases the lower bound c_0 if all unary costs $c_i(a)$ are non-zero.

Project(M,i,a,α)

Precondition: $i \in M$, $a \in d_i$, $-c_i(a) \leq \alpha \leq \min\{c_M(x): x[i]=a\}$

$c_i(a) := c_i(a) + \alpha$;

for all $x \in$ labelings(M) s.t. $x[i]=a$ **do**

$c_M(x) := c_M(x) - \alpha$;

If $\alpha>0$, this projects costs from c_M to c_i

If $\alpha<0$, this extends costs from c_i to c_M

Node and soft arc consistency

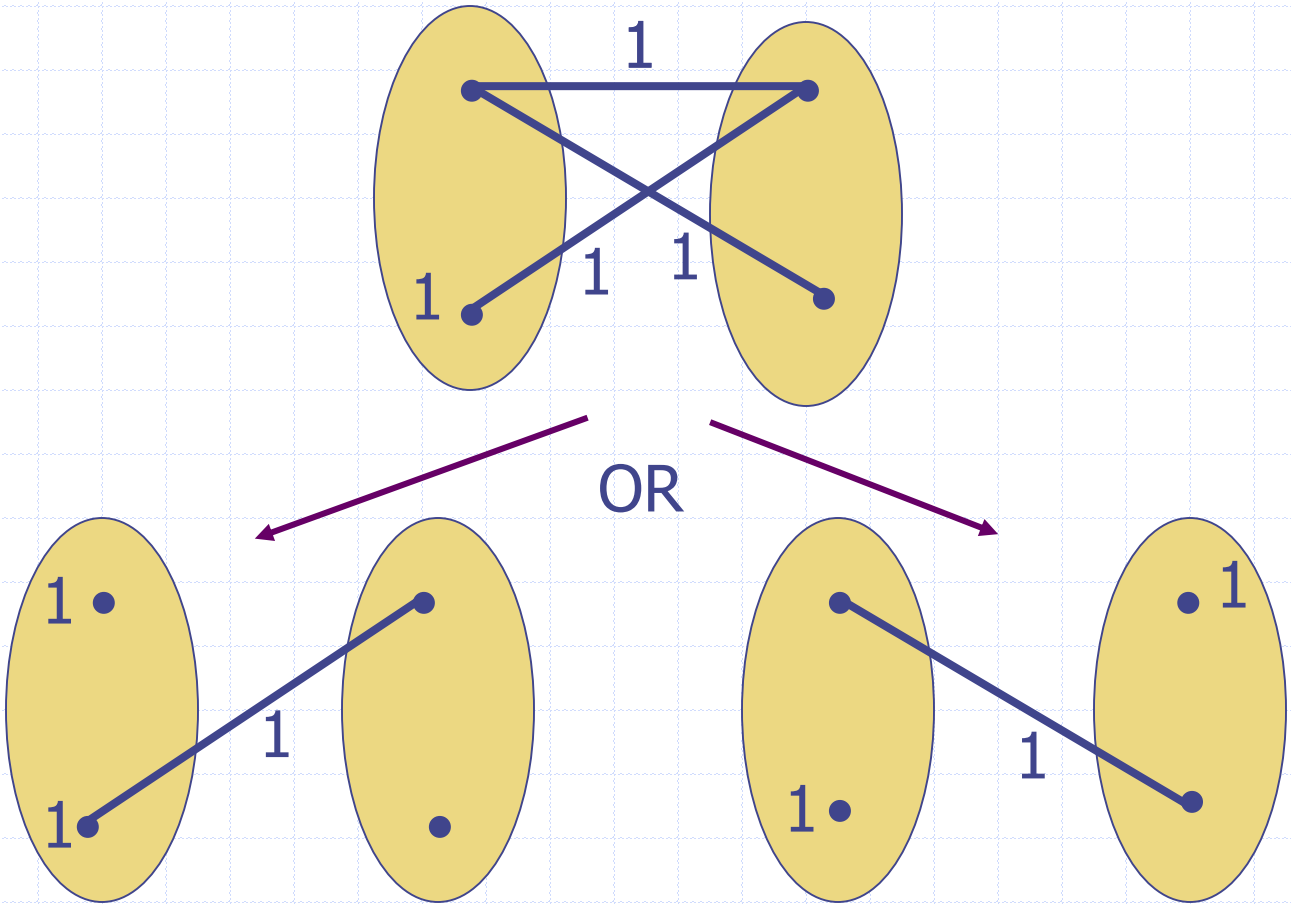
- Node consistent (NC) if $\forall i$

  no UnaryProject(i,α) is possible for $\alpha>0$ and no propagation of ∞ costs possible between c_i and c_0 (value $a \in d_i$ removed if $c_i(a)+c_0 \geq UB$)

- Soft arc consistent (SAC) if $\forall M,i,a$

  no Project(M,i,a,α) is possible for $\alpha>0$

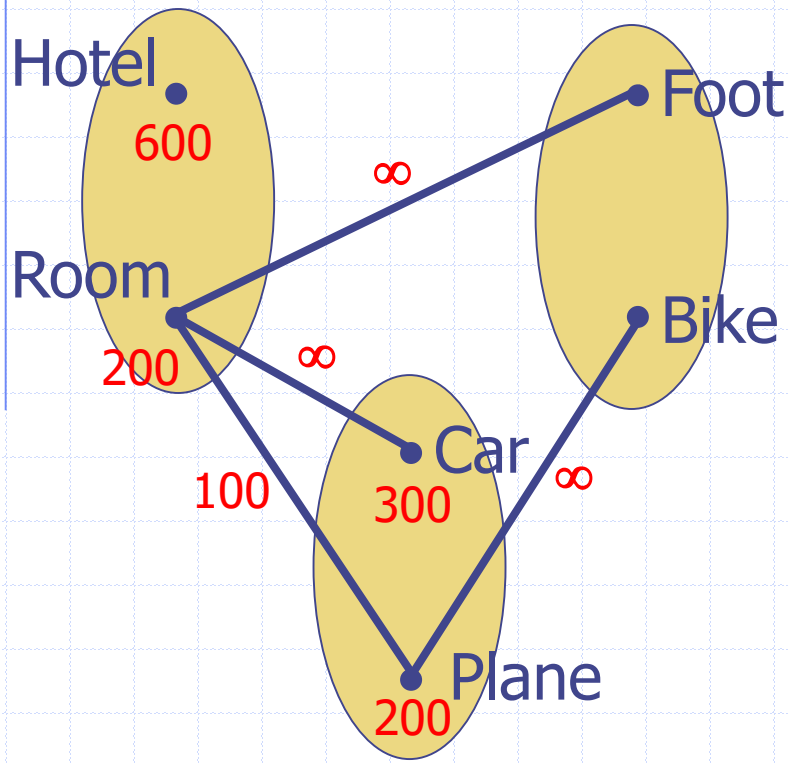
Soft AC closure is not unique



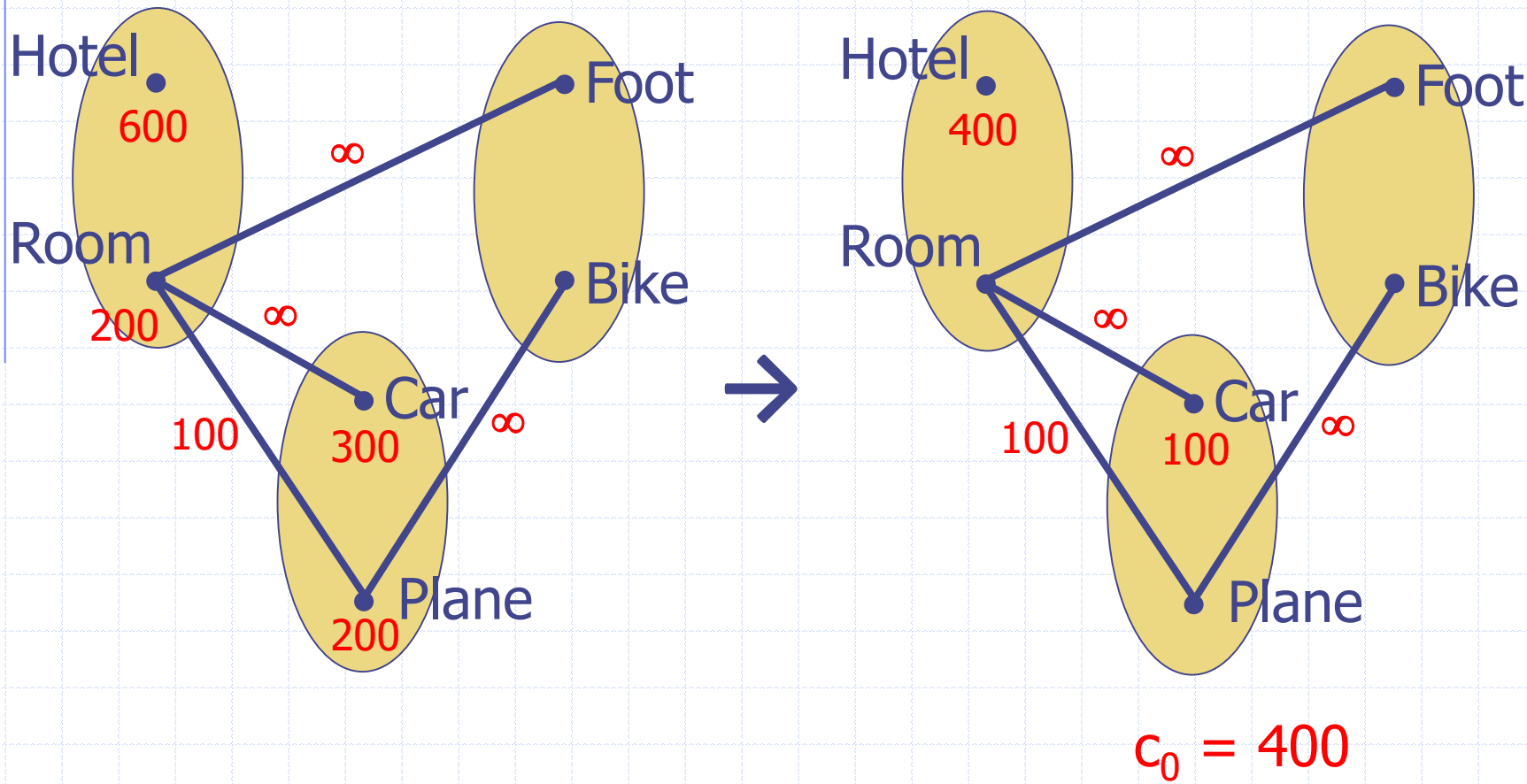
Different soft AC notions:

- Directional: send costs from X_j to X_i if $i<j$ (in the hope that this will increase c_0)
- Existential: $\forall i$, send costs to X_i simultaneously from its neighbor variables if this increases c_0
- Virtual: no *sequence* of Projects/Extends increases c_0
- Optimal: no *simultaneous set* of Projects/Extends increases c_0

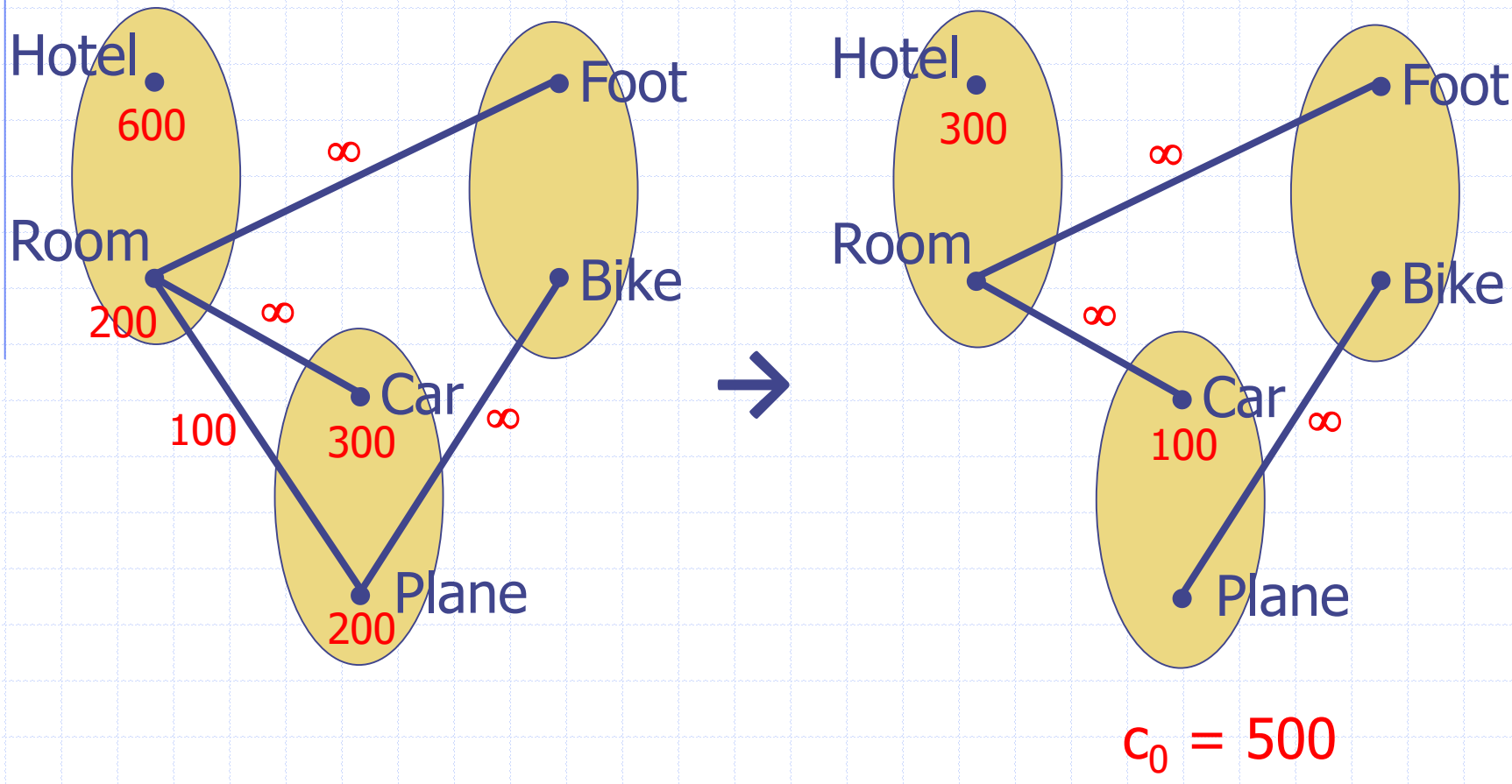
Example instance



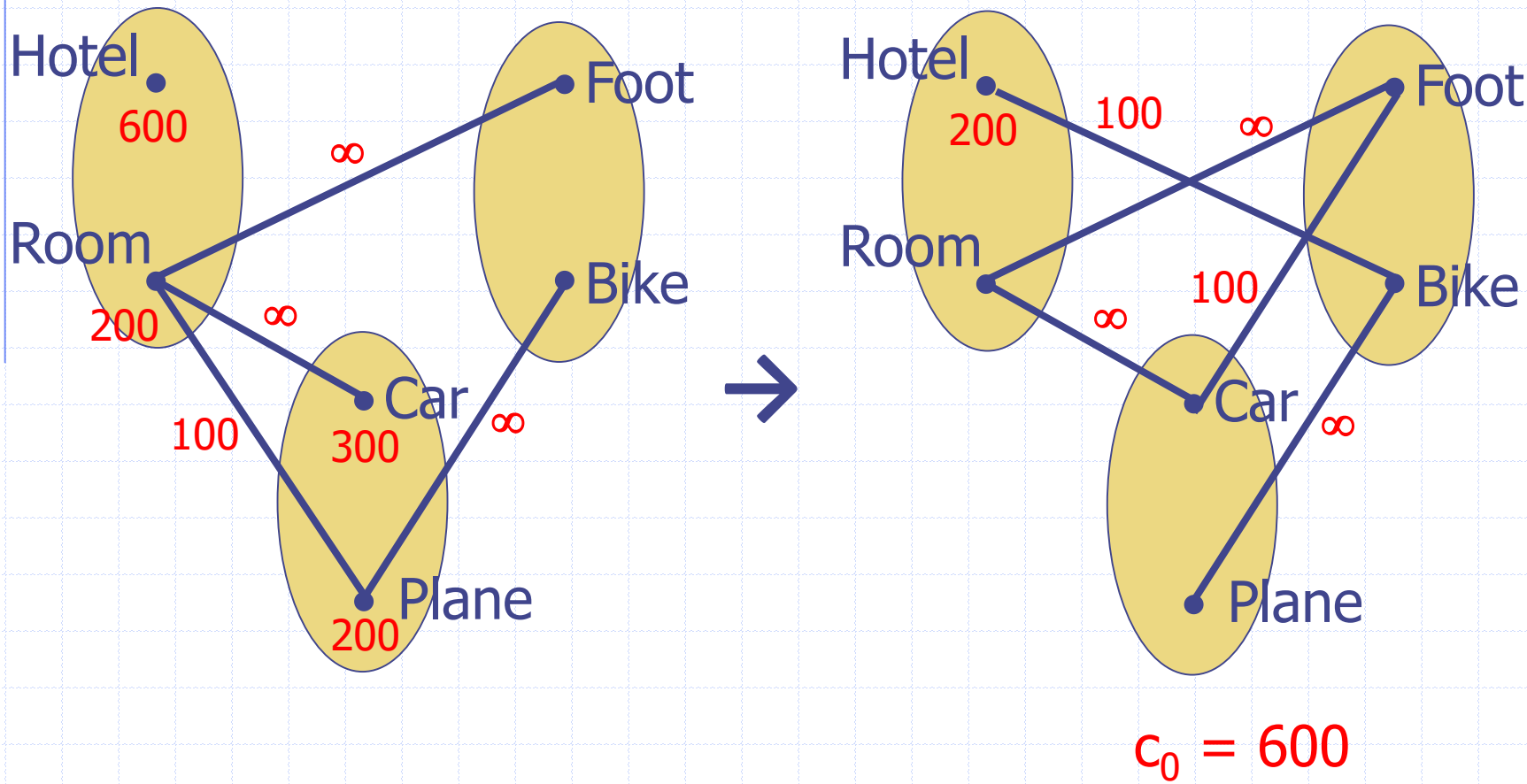
Example instance after NC



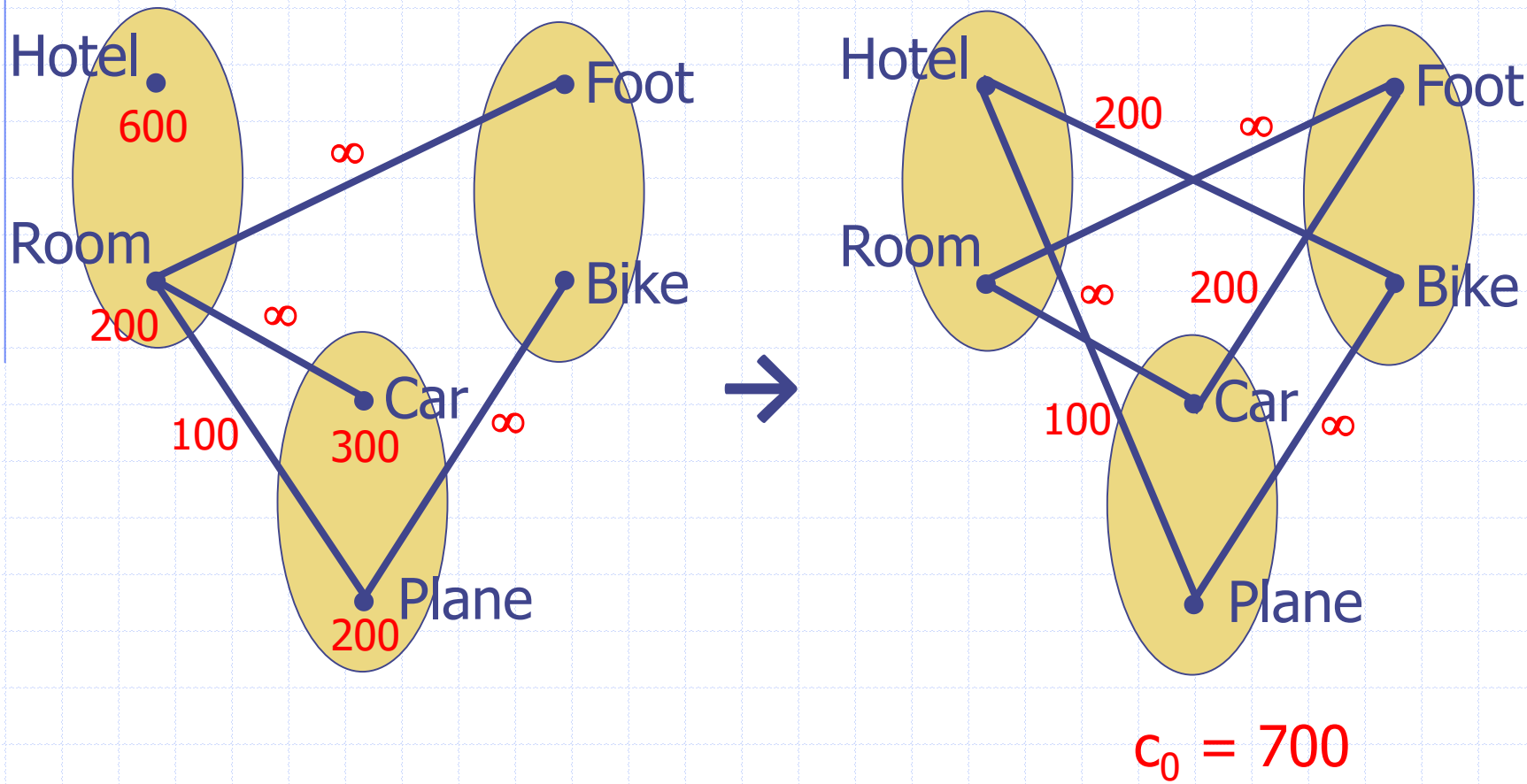
Example instance after SAC



Example instance after EDAC



Example instance after VAC



Directional Arc Consistency

- for all $i<j$, $\forall a \in d_i \; \exists b \in d_j$ such that $c_{ij}(a,b) = c_j(b) = 0$.
- Solves tree-structured CFNs
- DAC can be established in $O(ed^2)$ time

Existential Arc Consistency

- node consistent and $\forall i, \exists a \in d_i$ such that $c_i(a) = 0$ and for all cost functions c_{ij} , $\exists b \in d_j$ such that $c_{ij}(a, b) = c_j(b) = 0$
- EDAC = Existential AC + Directional AC + Soft AC
- EDAC can be established in $O(ed^2 \max\{nd, UB\})$ time

Virtual Arc Consistency (VAC)

- If P is a CFN instance then Bool(P) is the CSP instance whose allowed tuples are the zero-cost tuples in P-c_0
- If Bool(P) has a solution, then P has a solution of cost c_0 (but usually Bool(P) has no solution)
- Definition: P is VAC if Bool(P) is AC.

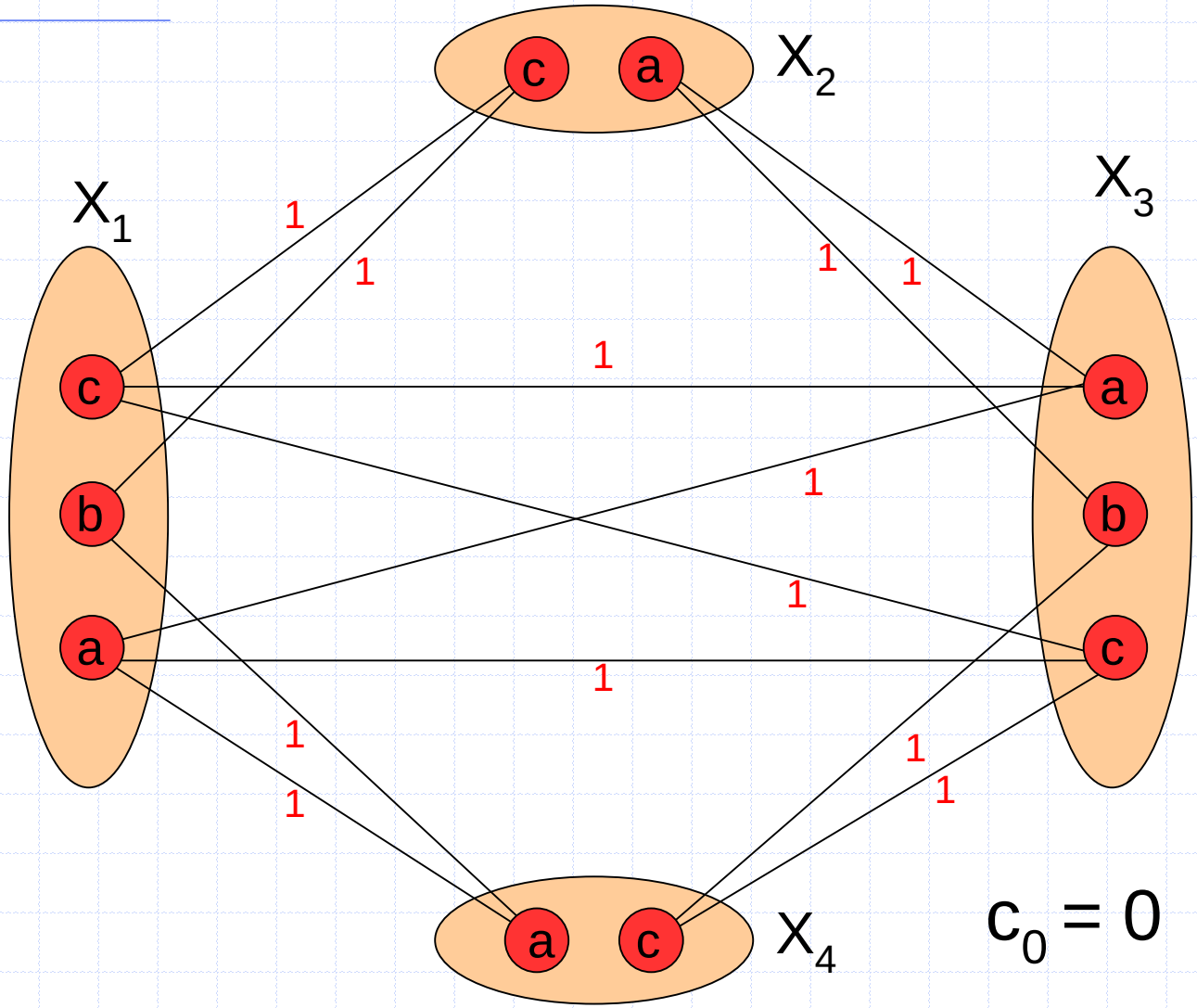
Approximating VAC

- If a sequence of AC operations in Bool(P) leads to a domain wipe-out, then a similar sequence of SAC operations in P increases c_0
- But, in this sequence, costs may need to be sent in more than one direction from the same $c_M \Rightarrow$ Introduction of *fractional* weights
- VAC algorithm may converge to a local minimum (an instance P' which is *not* VAC)

Optimal Soft Arc Consistency

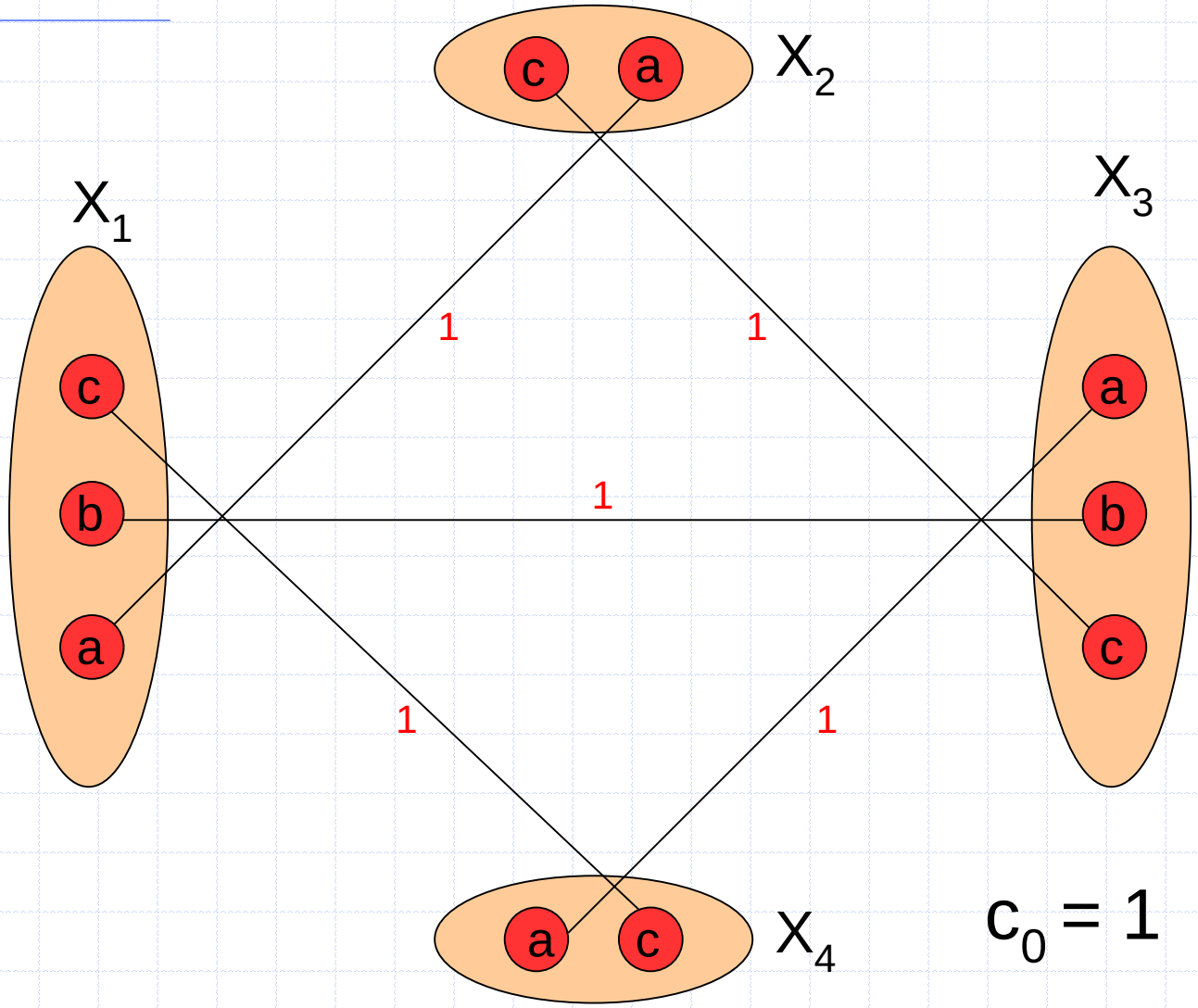
- We can overcome this problem of convergence by solving a LP to find the *set* of simultaneous UnaryProject and Project operations which maximises c_0.
- The resulting CFN instance is OSAC (Optimal Soft Arc Consistent).
- OSAC is strictly stronger than VAC.
- Unfortunately, the LP has O(edr+n) variables and O(ed^r+nd) constraints, and hence only useful for pre-processing.

Example of OSAC



$$p_{2c}^{23} = p_{3a}^{34} = p_{3b}^{31} = p_{1a}^{12} = p_{1c}^{14} = -1$$

$$p_{3a}^{23} = p_{3b}^{23} = p_{4c}^{34} = p_{1a}^{31} = p_{1c}^{31} = p_{2c}^{12} = p_{4a}^{14} = u_4 = 1$$



$c_0 = 1$

OSAC: Theoretical results

The value c_0 returned by OSAC is equal to the minimum expected cost of a locally (but not necessarily globally) coherent probabilistic assignment (= dual of LP)

E.g. 2-colouring on 3 variables: the probabilistic assignment

$$Pr_i(0) = Pr_i(1) = 0.5, \quad Pr_{ik}(01) = Pr_{ik}(10) = 0.5$$

is *locally* but not *globally* coherent and has expected cost 0.

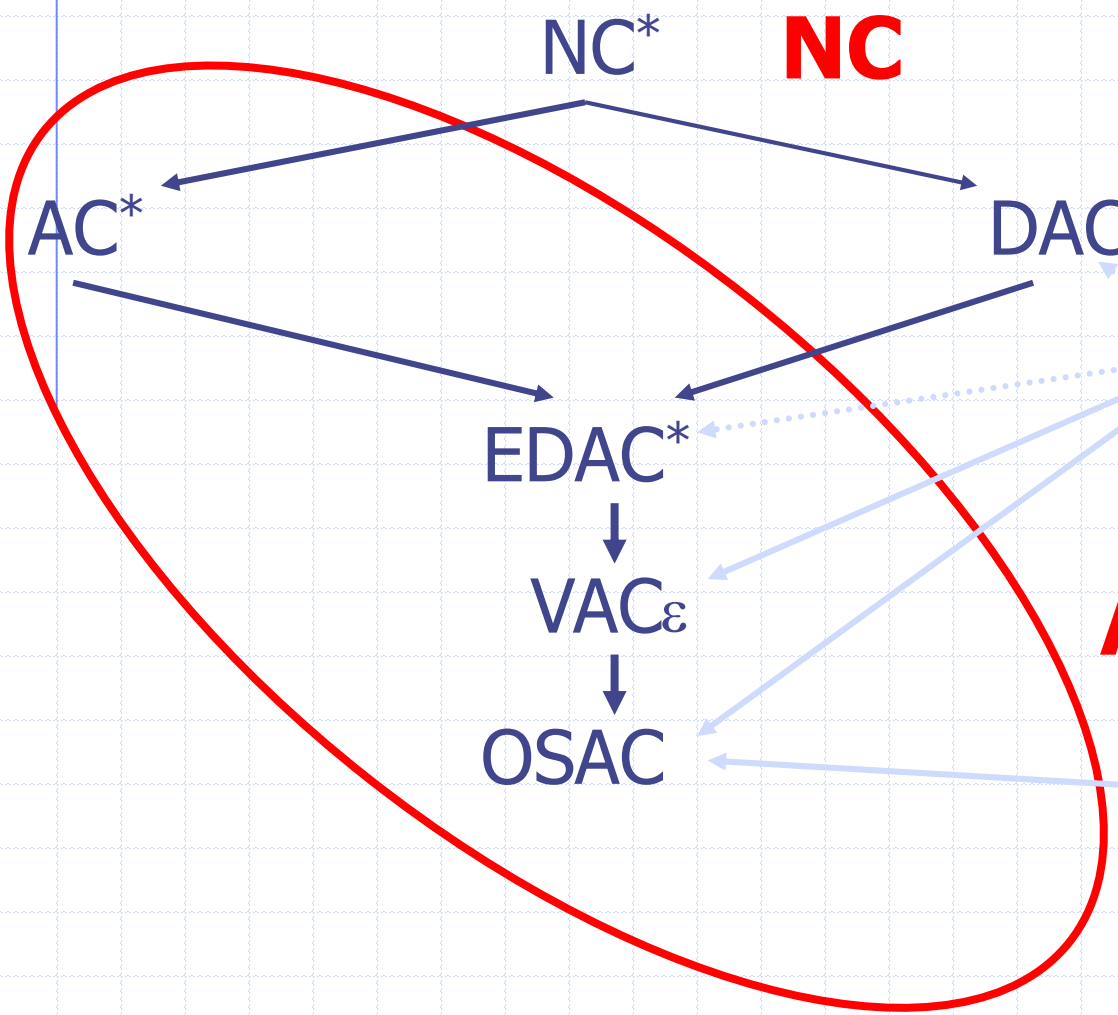
See M. Schlesinger, "Sintaksicheskiy analiz dvumernykh zritelnikh signalov v usloviyakh pomekh" (Syntactic analysis of two-dimensional visual signals in noisy conditions), *Kibernetika* (1976).

OSAC: Theoretical results

OSAC solves all tractable languages of finite-valued cost-functions (including submodular functions)

Hierarchy

Special case: CSP



NC

DAC

Solve tree-like primal graphs

AC

Solves all finite -valued tractable languages

Some practical observations

- For very hard-to-solve instances, maintaining VAC provides a significant speed-up, however for many problems, maintaining a simpler form of soft arc consistency (EDAC) is faster.
- Unary costs $c_i(a)$ useful for value and variable ordering heuristics.
- Consistencies can be generalised to abitrary arities but computational cost is high.

AC for soft global constraints

(See van Hoeve et al, *J. Heur.* 2006, Lee & Leung, *IJCAI* 2009, Allouche et al. AIJ 2016)

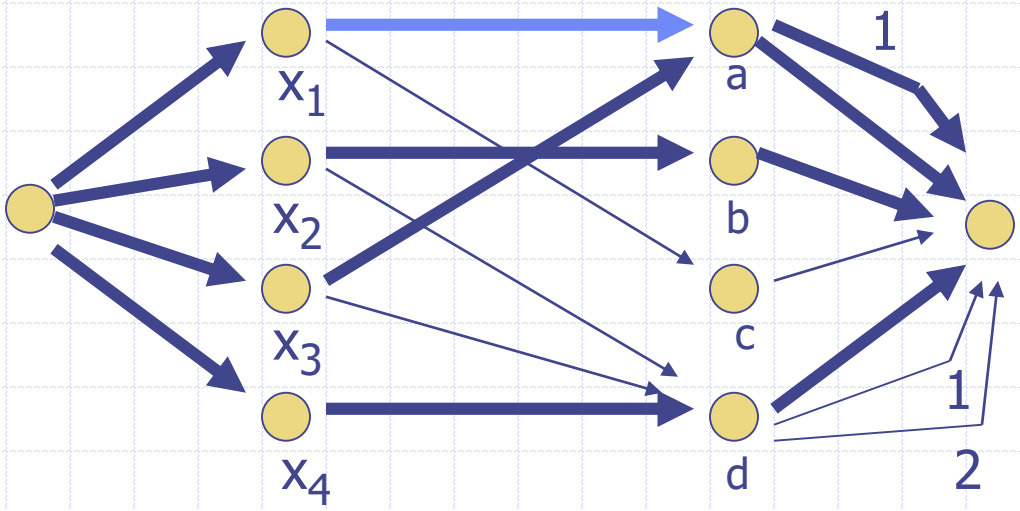
- Recent research has shown that many global cost functions allow projection/extension to/from unary cost functions in poly-time.
- Examples: cost-function variants of AMONG, ALLDIFFERENT, GCC, GRAMMAR, REGULAR, SAME, SUM...

Example: AC for flow-based soft global constraints

- Suppose that a global cost function c_M can be coded as the minimum cost of a maximum flow in a network in which (a) there is a one-to-one correspondence between max-flows and global labelings and (b) each assignment (x_i,a) has a corresponding edge e_{ia} such that the max-flow is 1 in e_{ia} if $x_i=a$ (0 if $x_i \neq a$).
- Then it is possible to project α from c_M to $c_i(a)$ by reducing cost(e_{ia}) by α.

Network representing soft Alldiff

- Min number of variables with same value
  variable-based costs (Beldiceanu & Petit, CPAIOR'04)



All edge capacities are equal to 1

All edge costs are 0 if not indicated

The flow shown is a min-cost max-flow with x_1=a. We can project 1 from c_M to c_1(a) by reducing the cost of the light blue edge from 0 to –1.

Decomposition of global cost functions

- Rewrite the global cost function as a sum of smaller bounded scope cost functions (a sub-network) *(Schiex et al, AAAI 2012)*
- Polynomial transformation solved by DAC for AMONG, REGULAR, *SUM*…
- DAC and VAC solves Berge-acyclic decomposed networks

softRegular

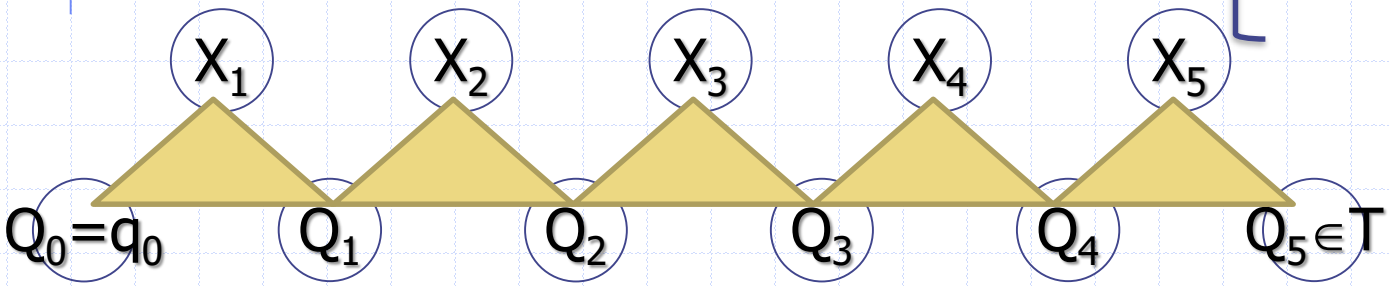
(*Hamming distance*)

softRegular(X_1, X_2, X_3, X_4, X_5, (Q, Σ, δ, q_0, T))

$$g(Q_{i-1}, X_i, Q_i) = \begin{cases} 0 & \text{if } (q_{i-1}, x_i, q_i) \in \delta \\ 1 & \text{if } (q_{i-1}, v, q_i) \in \delta, v \neq x_i \\ +\infty & \text{otherwise} \end{cases}$$

$D_{Xi} \in Q$

$D_{Qi} \in \Sigma$



➔ *Berge-acyclic decomposition*

1-softRegular

n	$\|\Sigma\|$	$\|Q\|$	Monolithic		Decomposed	
			filter	solve	filter	solve
25	5	10	0.12	0.51	0.00	0.00
		80	2.03	9.10	0.08	0.08
25	10	10	0.64	2.56	0.01	0.01
		80	10.64	43.52	0.54	0.56
25	20	10	3.60	13.06	0.03	0.03
		80	45.94	177.5	1.51	1.55
50	5	10	0.45	3.54	0.00	0.00
		80	11.85	101.2	0.17	0.17
50	10	10	3.22	20.97	0.02	0.02
		80	51.07	380.5	1.27	1.31
50	20	10	15.91	100.7	0.06	0.07
		80	186.2	1,339	3.38	3.47

toulbar2 version 0.9.5

Clique cuts

- Given a set S

$$x_i + x_j \leq 1 \quad \forall x_i, x_j \in S$$

$\Rightarrow$ Satisfied by $x_i = 0.5$

But we can get

$$\sum_{x_i \in S} x_i \leq 1$$

Clique cuts in CFN

(CP17)

Straightforward generalization
Given a set S of $\langle X_i, v_i \rangle$ with

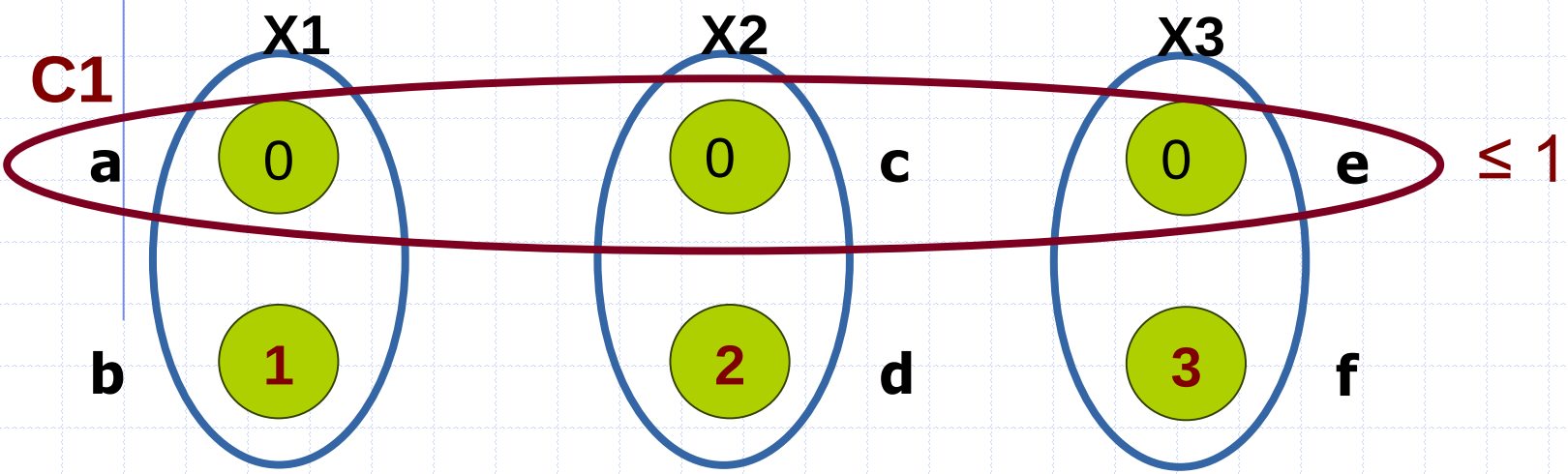
- $c_{ij}(v_i, v_j) = \infty$

Then derive

$$\sum_{ij \in S} x_{ij} \leq 1$$

Reparameterization for clique

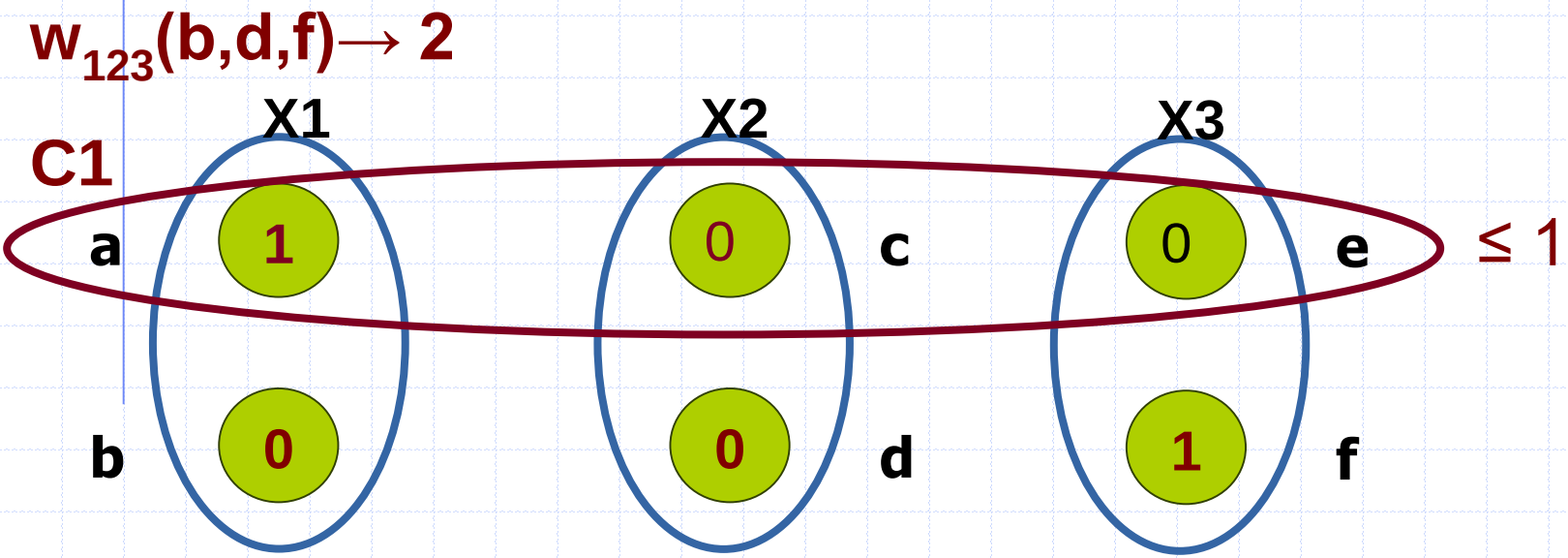
(CP17)



$w_{\varnothing}=0$

Reparameterization for clique

(CP17)



$w_{\emptyset}=3$

Experimental Results

(CP17)

problem	TOULBAR2		TOULBAR2^{clq}		CPLEX	
	solv.	time	solv.	time*	solv.	time
Auction/path	**86**	59	**86**	0.18	**86**	0.01
Auction/sched	**84**	110	**84**	0.23	**84**	0.04
MaxClique	31	1871	37	1508	**38**	**1533**
SPOT5	4	2884	6	2603	**16**	738

* Including bounded clique detection with Bron-Kerbosch algorithm in preprocessing

Bibliography

- For an overview of soft local consistencies, see "*Soft arc consistency revisited*", Cooper, de Givry, Sanchez, Schiex, Zytnicki & Werner, *AIJ* 2010.

- For soft global constraints (flow, dp, dec), see "*Tractability-preserving transformations of global cost functions*", Allouche *et al*, *AIJ* 2016.

- "*Clique Cuts in Weighted Constraint Satisfaction*", Katsirelos et al, *CP2017*.