A semiring-based approach to multi-objective optimization

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Abstract. Many combinatorial optimization problems require the assignment of a set of variables in such a way that an objective function is optimized. Most often, the objective function itself involves different criteria, and it may happen that the single requirements are in conflict: assignments that are good with respect to one objective may behave badly with respect to another. Thus, an optimal solution with respect to all criteria may not exist, and either the efficient frontier (the set of best incomparable solutions, all equally relevant in the absence of further information) or an approximation has to be looked after. Indeed, computing the efficient frontier should be preferred over computing an approximation.
The paper shows how the soft constraints formalism based on semirings, so far exploited for finding approximations, can embed also the computation of the efficient frontier in multi-objective optimization problems. The main result is the proof that the efficient frontier of a multi-objective problem can be obtained as the so called best level of consistency distilled from a suitable soft constraint problem.

1 Introduction

Many real world problems involve multiple measures of performance, or objectives, that should be optimized simultaneously: see e.g. the survey [4] and the references therein. In such a situation an unique, perfect solution may not exist, while a set of solutions can be found that should be considered equivalent in the absence of information concerning the relevance of each objective wrt the others. Hence, two solutions are equivalent if one of them is better than the other for some criteria, but possibly worse for others; while one solution dominates (is better than) the other if the former is better than the latter for all criteria.

The set of best solutions is the set of efficient (or pareto-optimal) solutions. The set of costs associated to efficient solutions is called the efficient frontier. The main task in a multi-objective problem is to compute the efficient frontier (and, possibly, one efficient solution for any of its elements).

The main goal of the paper is to prove that the computation of the efficient frontier of a multi-objective optimization problems can be modeled using soft CSP. More precisely, our main contribution is to show, given a (possibly partially ordered) semiring $\mathcal{K}$, how to compute a new semiring $\mathcal{I}(\mathcal{K})$ such that its elements corresponds to sets of (irreducible) elements of the original semirings; and such that the set of optimal costs of the original problem corresponds to the optimal solution computed in the derived semiring.

When applied to the multi-objective context, the result of our work can be summarized as follows: consider a multi-objective problem where $\mathcal{K}_1, \mathcal{K}_2 \ldots \mathcal{K}_p$ are the semirings associated to each objective. If we use their cartesian product $\mathcal{K}_C = \mathcal{K}_1 \times \ldots \times \mathcal{K}_p$ to model the multi-objective problem, the solution corresponds to the lowest vector that dominates the efficient frontier. Further, if we use $\mathcal{I}(\mathcal{K}_C)$ to model the problem, the solution coincides with the efficient frontier.

2 Remarks on optimization

Many combinatorial optimization problems are defined in terms of a set of decision variables V, a (finite) domain of interpretation D, and an objective function $F : (V \to D) \to A$. The objective function F associates an *outcome* in A to each variable assignment $\eta : V \to D$.

If A is ordered, we may consider suitable notions of optimality. If A is totally ordered, the optimum of the objective function (assuming optimization as maximization) is

$$\max_{\eta}\{F(\eta)\}$$

The optimum is the highest possible outcome given by the assignments. However, often the set A is only partially ordered. Then, it is customary to characterize a set of optima

$$\mathcal{E} = \{F(\eta) \mid \forall \eta'. F(\eta') \not> F(\eta)\}$$

Elements in $\mathcal{E}$ are optimal in the sense that no better values exist for objective F, and they are pairwise uncomparable. Since computing $\mathcal{E}$ can be very resource consuming, an alternative is simply to consider an approximation raising from the objective function, namely, $\sup_{\eta}\{F(\eta)\}$.

Arguably, the most frequent case of problems involving a partially ordered set A concerns multi-objective optimization. The set of outcomes $A = A_1 \times \ldots \times A_p$ is a p-dimensional space where each component A_i is a (totally) ordered set associated to one criteria. Given two values $\boldsymbol{a} = \langle a_1, a_2 \ldots a_p \rangle, \boldsymbol{b} = \langle b_1, b_2 \ldots b_p \rangle \in A$, the usual partial order is called *dominance*, and it is defined as $\boldsymbol{a} \leq \boldsymbol{b}$ iff $\forall i \in \{1, 2 \ldots p\}.\ a_i \leq b_i$. In that context, it is usually important to compute the set $\mathcal{E}$, which is called the *efficient frontier* of the problem. Those assignments η such that $F(\eta) \in \mathcal{E}$ are called *pareto-optimal* solutions.

3 On semiring-based frameworks

Semirings provide an algebraic framework for the specification of a general class of combinatorial optimization problems. Outcomes associated to variable instantiations are modeled as elements of a set A, equipped with a sum and a product operator. These operators are used for combining constraints: the intuition is that the sum operator induces a partial order $a \leq b$, meaning that b is a better outcome than a; whilst the product operator denotes the aggregation of outcomes coming from different soft constraints.

3.1 The algebra of semirings

In this section we review the main algebraic concepts. Since there seems to be no converging definition in the literature for *semirings*, this section chooses a minimal approach, and briefly presents the main notions concerning that algebraic structure that is going to be used later on in the presentation of the soft constraints approach. We adopt a terminology inspired by [5] and, in lesser degree, by [7], aiming at a smooth presentation of the main concepts.

Definition 1 (semirings). *A (commutative) semiring is a five-tuple*

$$\mathcal{K} = \langle A, +, \times, \mathbf{0}, \mathbf{1} \rangle$$

such that A is a set, $\mathbf{1}, \mathbf{0} \in A$, and $+, \times : A \times A \rightarrow A$ are binary operators making the triples $\langle A, +, \mathbf{0} \rangle$ and $\langle A, \times, \mathbf{1} \rangle$ commutative monoids (semigroups with identity), satisfying

- $\forall a, b, c \in A . a \times (b + c) = (a \times b) + (a \times c)$;
- $\forall a \in A . a \times \mathbf{0} = \mathbf{0}$.

A semiring is absorptive if additionally $\forall a \in A . a + \mathbf{1} = \mathbf{1}$.

Absorptive semirings are also known as *simple*, and the property is equivalent to $\forall a, b \in A . a + (a \times b) = a$, that is, each element $a \times b$ is "absorbed" by a. These semirings represent the structure we put at the base of our proposal since the three properties they satisfy (absorptiveness, zero and unit element) seem pivotal for soft constraint frameworks.

We can now state a simple characterization result linking absorptiveness to idempotency and to a top element.

Proposition 1. *Let $\mathcal{K}$ be an absorptive semiring. Then, the sum operator $+$ of $\mathcal{K}$ is idempotent.*

The former result is well-know in the literature, and semirings such that the sum operator is idempotent are called *dioids* or *tropical semirings*. These structures are well-studied in the literature(see e.g. the references in [7]), and we take advantage of classical constructions in the following sections.

Proposition 2. *Let $\mathcal{K} = \langle A, +, \times, \mathbf{0}, \mathbf{1} \rangle$ be a tropical semiring. Then, the relation $\langle A, \leq \rangle$ such that $\forall a, b \in A . a \leq b$ iff $a + b = b$ is a partial order.*

Moreover, if $\mathcal{K}$ is absorptive, then $\mathbf{1}$ is the top element of the partial order. If additionally $\mathcal{K}$ is absorptive and *idempotent* (that is, the product operator $\times$ is idempotent), then the partial order is actually a *lattice*, since $a \times b$ corresponds to the greatest lower bound of a and b.

3.2 Soft constraints based on semirings

This section briefly recalls the main concepts of the semiring-based approach to the soft CSPs framework.

Definition 2 (constraints). *Let $\mathcal{K} = \langle A, +, \times, \mathbf{0}, \mathbf{1}\rangle$ be an absorptive semiring; let V be a set of variables; and let D be a finite domain of interpretation for V. Then, a* constraint *$(V \to D) \to A$ is a function associating a value in A to each assignment $\eta : V \to D$ of the variables.*[5]

Note that even if a constraint involves all the variables in V, it must depend on the assignment of a finite subset of them. For instance, a binary constraint $c_{x,y}$ over variables x, y is a function $c_{x,y} : (V \to D) \to A$ which depends only on the assignment of variables $\{x, y\} \subseteq V$. We call this subset the *support* of the constraint [3] and correspond to the classical notion of scope of a constraint. Most often, whenever V is ordered, an assignment (over a support of cardinality k) is concisely presented by a tuple in D^k.

Definition 3 (support). *Let $c \in \mathcal{C}$ be a constraint. Its support $supp(c)$ is the set $\{v \in V \mid \exists \eta, d_1, d_2 . c\eta[v := d_1] \neq c\eta[v := d_2]\}$, where*

$$\eta[v := d]v' = \begin{cases} d & \textit{if } v = v' \\ \eta v', & \textit{otherwise} \end{cases}$$

Note that $c\eta[v := d_1]$ means $c\eta'$ where η' is η modified with the assignment $v := d_1$ (that is, the operator $[\_ := \_]$ has precedence over application). Note also that $c\eta$ is the application of a constraint function $c : (V \to D) \to A$ to a function $\eta : V \to D$, obtaining a semiring value.

Combining and projecting soft constraints

Definition 4 (combination). *The* combination *operator $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is defined as $(c_1 \otimes c_2)\eta = c_1\eta \times c_2\eta$.*

Thus, combining two constraints means building a new constraint whose support involves all the variables of the original ones (i.e., $supp(c_1 \otimes c_2) \subseteq supp(c_1) \cup supp(c_2)$), and which associates to each tuple for such variables a semiring element, obtained by multiplying the elements associated by the original constraints to the appropriate subtuples.

[5] Alternatively, a constraint is a pair $\langle con, def\rangle$: con is the scope of a constraint, and def the function associating a value in A to each assignment of the variables in con.

Definition 5 (projection). *Let $c \in \mathcal{C}$ be a constraint and $v \in V$ a variable. The* projection *of c over $V - \{v\}$ is the constraint c' such that $c'\eta = \sum_{d \in D} c\eta[v := d]$.*

We denote such projection as $c \Downarrow_{(V-\{v\})}$. The projection operator can be inductively extended to a set of variables $I \subseteq V$ by $c \Downarrow_{(V-I)} = c \Downarrow_{(V-\{v\})} \Downarrow_{(V-\{I-\{v\}\})}$. Informally, projecting means eliminating variables from the support.

Soft CSPs, Solutions and optimizations

Definition 6 (soft CSPs). *A* soft constraint satisfaction problem *is a pair $\langle C, con \rangle$, for C is a set of constraints over variables $con \subseteq V$.*

The set *con* is the set of variables of interest for the constraint set C, which may concern also variables not in *con*.

Definition 7 (solutions). *The* solution *of a soft CSP $P = \langle C, con \rangle$ is the constraint $Sol(P) = (\bigotimes C) \Downarrow_{con}$.*

The solution of a soft CSP is obtained by combining all constraints, and then projecting over the variables in *con*. In this way we get the constraint with support (not greater than) *con* which is "induced" by the entire soft CSP.

What is called the solution of a soft CSP plays indeed the role of the (implicit) objective function in optimization problems. We may now start referring to the efficient frontier $\mathcal{E}(P)$ of a soft CSP: pareto-optimal solutions are referred to as *abstract solutions* in the soft CSP literature. We may also refer now to best approximation, which may now be neatly characterized by so-called best level of consistency.

Proposition 3. *Let $P = \langle C, con \rangle$ be a soft CSP, and $blevel(P) = (\bigotimes C) \Downarrow_{\emptyset}$ be denoted as the* best level of consistency *of P. Then, $sup_\eta\{Sol(P)(\eta)\} = blevel(P)$.*

Some instantiations The interest of the soft CSP framework is that it can accomodate several soft constraint frameworks by just instantiating the absortpive semiring. For instance,

- Classical CSPs are soft CSPs where the semiring is

$$\mathcal{K}_{CSP} = \langle \{false, true\}, \vee, \wedge, false, true \rangle$$

  The induced order is $false < true$. The consistency function is $F(\eta) = \bigwedge_{c \in C} c\eta$ (i.e, $F(\eta) = true$ iff η satisfies all the constraints). The best level of consistency is $blevel(P) = \bigvee_\eta F(\eta)$ (i.e. $blevel(P) = true$ iff at least one assignment satisfies all the constraints)
- Weighted CSPs are soft CSPs where the semiring is

$$K_{WCSP} = \langle \mathcal{R}, min, +, \infty, 0 \rangle$$

  The induced order is the usual order among reals. The objective function is $F(\eta) = \sum_{c \in C} c\eta$ (i.e, $F(\eta)$ is the sum of values given by all the constraints to η). The best level of consistency is $blevel(P) = \min_\eta\{F(\eta)\}$.

The two semirings above are totally ordered, so $blevel(P)$ coincides with the optimum of the problem. That does not hold for partially ordered semirings, as next section shows.

Figure 1 graphically represents a weighted CSP. Variables and constraints are nodes and undirected (unitary for c_1 and c_3 and binary for c_2) arcs, respectively, and semiring values are written to the right of the corresponding tuples. The variables of interest (the set *con*) are represented with a double circle. Here we assume that the domain D of the variables contains only elements a, b and c.

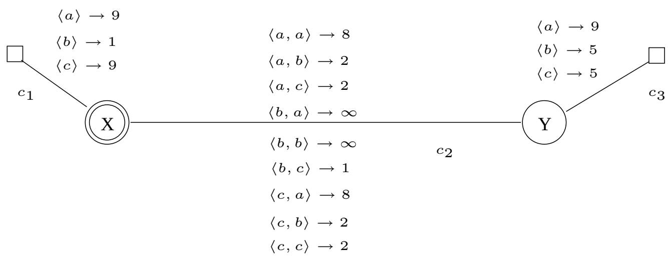

Fig. 1. A weighted CSP.

Note that $Sol(P)$ has support x and is $Sol(P)(a) = 16$, $Sol(P)(b) = 7$ and $Sol(P)(c) = 16$, while the optimum of P is $blevel(P) = 7$.

3.3 Partial order and cartesian product

Partially ordered semirings naturally arise whenever multi-objective optimization problems are of interest. Indeed, it is easy to show how the cartesian product $\mathcal{K}_C$ of a family $\mathcal{K}_1, \ldots, \mathcal{K}_p$ of semirings (each one associated to an objective function) is also equipped with a semiring structure, where the sum and product operators are defined pointwise.

Proposition 4 (cartesian product semirings). *Let* $\{\mathcal{K}_i = \langle A_i, +_i, \times_i, \mathbf{0}_i, \mathbf{1}_i \rangle\}_{1 \leq i \leq p}$ *be a family of semirings, and* $\mathcal{K}_C = \langle A, +, \times, \mathbf{0}, \mathbf{1} \rangle$ *their cartesian product, defined as*

- $A = A_1 \times A_2 \times \cdots \times A_p$
- $\forall_{\boldsymbol{v},\boldsymbol{w} \in A},\ \boldsymbol{v} + \boldsymbol{w} = \langle v_1 +_1 w_1, v_2 +_2 w_2, \ldots, v_p +_p w_p \rangle$
- $\forall_{\boldsymbol{v},\boldsymbol{w} \in A},\ \boldsymbol{v} \times \boldsymbol{w} = \langle v_1 \times_1 w_1, v_2 \times_2 w_2, \ldots, v_p \times_p w_p \rangle$
- $\mathbf{0} = \langle \mathbf{0}_1, \mathbf{0}_2, \ldots, \mathbf{0}_p \rangle$
- $\mathbf{1} = \langle \mathbf{1}_1, \mathbf{1}_2, \ldots, \mathbf{1}_p \rangle$

Then, $\mathcal{K}_C$ *is a semiring. Moreover, if each* $\mathcal{K}_i$ *is tropical (absorptive, idempotent), then also* $\mathcal{K}_C$ *is so.*

The result is standard in soft CSP literature. Note that the order induced by the semiring $\mathcal{K}_C$ corresponds to the notion of dominance used in multi-objective optimization.

Corollary 1. *Let $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{K}_C$. Then, $\boldsymbol{v} \leq_{\mathcal{K}_C} \boldsymbol{w}$ iff for each $i \in \{1 \dots p\}$ we have $v_i \leq_{\mathcal{K}_i} w_i$.*

From Proposition 3, the best level of consistency of a problem P over semiring $\mathcal{K}_C$ is the lowest vector dominating the efficient frontier $\mathcal{E}(P)$. However, we are interested in computing $\mathcal{E}(P)$ as the best level of consistency of some soft CSP problem. The following section is devoted to this issue.

4 Semirings based on powersets

This section states the main theorem of the paper, namely, that for each soft problem P over a semiring $\mathcal{K}$, a new semiring $\mathcal{I}(\mathcal{K})$ and a semiring morphism $i : \mathcal{K} \to \mathcal{I}(\mathcal{K})$ can be devised such that the best level of consistency for the modified problem $i(P)$ coincides with the efficient frontier of P.

For the sake of readability, in the rest of the section we fix a (commutative) semiring $\mathcal{K} = \langle A, +, \times, \mathbf{0}, \mathbf{1} \rangle$.

4.1 Partial correctness semiring

Let first introduce the notion of *closure* of a subset $X \subseteq A$:

Definition 8 (downward closure). *Let $\mathcal{K}$ be a tropical semiring. Then, for a set $S \subseteq A$ we let ΔS denote its* downward closure*, i.e., the set $\{a \in A \mid \exists s \in S.a \leq_{\mathcal{K}} s\}$.*

A set S is *downward closed* if $S = \Delta S$ (and any downward closure is so, since $\Delta(\Delta S) = \Delta S$), and we denote by $\mathcal{L}(A)$ the family of downward closed subsets of A.

Proposition 5. *Let $\mathcal{K}$ be an absorptive semiring. Then, the five-tuple $\mathcal{L}(\mathcal{K}) = \langle \mathcal{L}(A), \cup, \times_H, \{\mathbf{0}\}, A \rangle$ is an absorptive semiring: its elements are the (not empty) downward-closed subsets of A, $S \cup T$ is set (of subsets) union, and $S \times_H T = \Delta(S \times T)$, for $S \times T = \{s \times t \mid s \in S, t \in T\}$.*

The closure of $\{s \times t \mid s \in S, t \in T\}$ is necessary, since in general it is not downward-closed. Note that the absorptiveness of $\mathcal{K}$ plays a pivotal role, since it means that $A = \Delta\{\mathbf{1}\}$.

It is worthwhile to see the ordering associated to $\mathcal{L}(\mathcal{K})$.

Corollary 2. *Let $\Delta S, \Delta T \in \mathcal{L}(\mathcal{K})$. Then, $\Delta S \leq_{\mathcal{L}(\mathcal{K})} \Delta T$ iff for each $s \in S$ there exists $t \in T$ such that $s \leq_{\mathcal{K}} t$.*

This result tells us that our construction of $\mathcal{L}(\mathcal{K})$ is reminiscent of the so-called *partial correctness* (or *Hoare*) powerdomain (a well-known tool in the field of denotational semantics: see e.g. [6]), hence the name.

We now state the first main theorem of our paper.

Theorem 1. *Let $P = \langle C, con\rangle$ be a soft CSP over the semiring K; and let $\mathcal{L}(P) = \langle C', con\rangle$ be the soft CSP over the semiring $\mathcal{L}(\mathcal{K})$ such that $C' = \{i(c) \mid c \in C\}$, for $i(c)(\eta) = \{c(\eta)\}$. Then, $\Delta(\mathcal{E}(P)) = blevel(\mathcal{L}(P))$.*

The closure $\Delta(\mathcal{E}(P))$ is necessary, since the sets in $\mathcal{L}(\mathcal{K})$ are downward-closed. However, note that each constraint of a soft CSP problem P is defined only over a finite set of functions $V \to D$, since it is finitely supported and D is finite. Thus, the efficient frontier $(\mathcal{E}(P))$ is always a finite set, and thus we can improve on the previous representation.

4.2 On finite representations of closures

Let us define a set $S \in \mathcal{L}(A)$ to be *finitely* downward-closed if there exists a finite, not-empty set $T \in A$ such that $S = \Delta(T)$. It is easy to see that the family $\mathcal{L}_f(A)$ of such sets can be equipped with a semiring structure.

Proposition 6. *Let $\mathcal{K}$ be an absorptive semiring. Then, the five-tuple $\mathcal{L}_f(\mathcal{K}) = \langle \mathcal{L}_f(A), \cup, \times_F, \{\mathbf{0}\}, A\rangle$ is an absorptive semiring: its elements are the finitely downward-closed subsets of A, $\Delta S \cup \Delta T$ is set union, and $\Delta S \times_F \Delta T = \Delta(S \times T)$, for $S \times T = \{s \times t \mid s \in S, t \in T\}$.*

The semiring $\mathcal{L}_f(\mathcal{K})$ is well-defined, since $\Delta S \cup \Delta T = \Delta(S \cup T)$ and $\Delta(\Delta S \times \Delta T) = \Delta(S \times T)$ for any sets S, T. In fact, $\mathcal{L}_f(\mathcal{K})$ is a subsemiring of $\mathcal{L}(\mathcal{K})$: a proper one, since there exist subsets that are not finitely downward-closed.

Of course, there may exist finite sets S, T such that $S \neq T$ and $\Delta S = \Delta T$. More precisely, this is due to the fact that a set S may be *redundant*, i.e., it may contain elements s such that $\Delta(S - \{s\}) = \Delta S$. In general, it is impossible to distill from a set S with infinite elements an irredundant set X such that $S = \Delta X$, while it is always possible if S is finite.

Proposition 7. *Let $S \in \mathcal{L}_f(\mathcal{K})$, and let $lubs\, S = \{s \in S \mid \nexists t : t \in S \wedge s \leq_{\mathcal{K}} t\}$. Then, $lubs\, S$ is an irredundant set, and moreover $S = \Delta(lubs\, S)$.*

In fact, note that $lubs\, S$ is the *unique* irredundant set generating S. Thus, using Proposition 7, we may distill a semiring $\mathcal{I}(\mathcal{K})$, which is isomorphic (i.e., the inclusion is the semiring identity) to $\mathcal{L}_f(\mathcal{K})$, and whose elements belong to the family $\mathcal{I}(A)$ of finite, irreducible subsets of A.

Proposition 8. *Let $\mathcal{K}$ be an absorptive semiring. Then, the five-tuple $\mathcal{I}(\mathcal{K}) = \langle \mathcal{I}(A), +_I, \times_I, \{\mathbf{0}\}, \{\mathbf{1}\}\rangle$ is an absorptive semiring: its elements are the finite irreducible subsets of A, $S +_I T = lubs\,(S \cup T)$, and $S \times_I T = lubs\,(S \times T)$, for $S \times T = \{s \times t \mid s \in S, t \in T\}$.*

Proposition 9. *Let $\mathcal{K}$ be an absorptive semiring. Then, the function $\Delta : \mathcal{I}(\mathcal{K}) \to \mathcal{L}_f(\mathcal{K})$, mapping S to ΔS, is a semiring isomorphism, with inverse given as $lubs : \mathcal{L}_f(\mathcal{K}) \to \mathcal{I}(\mathcal{K})$, mapping S to $lubs\, S$.*

To prove that Δ is a semiring isomorphim boils down to show that $\Delta(S +_I T) = \Delta S \cup \Delta T$, and $\Delta(S \times_I T) = \Delta S \times_F \Delta T$. These are easily accomplished, after noting that by definition $\Delta T = \Delta(lubs\, T)$ and $lubs\,(\Delta(lubs\, T)) = lubs\, T$.

We may now state the main theorem of our paper.

Theorem 2. *Let $P = \langle C, con\rangle$ be a soft CSP over the semiring K; and let $\mathcal{I}(P) = \langle C', con\rangle$ be the soft CSP over the semiring $\mathcal{I}(\mathcal{K})$ such that $C' = \{i(c) \mid c \in C\}$, for $i(c)(\eta) = \{c(\eta)\}$. Then, $\mathcal{E}(P) = blevel(\mathcal{I}(P))$.*

The proof exploits the finiteness of the efficient frontier $\mathcal{E}(P)$, in order to prove that $\Delta(\mathcal{E}(P)) = blevel(\mathcal{L}_f(P))$, and finally Proposition 9, noting that $\mathcal{E}(P)$ is irredundant.

4.3 Remarks on local consistency

One of the appealing aspects of the soft CSP framework is the development of generic local consistency algorithms, which are employed to distill the best level of consistency of a problem P, irregardless of the semiring at hand.

The following proposition is pivotal for extending the algorithms of local consistency.

Proposition 10. *Let $\mathcal{K}$ be an absortpive semiring. If $\mathcal{K}$ is idempotent, then also both $\mathcal{L}(\mathcal{K})$ and $\mathcal{I}(\mathcal{K})$ are so.*

Hence, the local consistency techniques applied for the soft problems over an idempotent semiring $\mathcal{K}$ can still be applied for problems over $\mathcal{I}(\mathcal{K})$.

4.4 Summing up Section 4

This section contains the theoretical novelties of the paper. The first result is the proof that for any absorptive semiring $\mathcal{K}$, the semiring $\mathcal{L}(\mathcal{K})$ of its downward-closed subsets is also absorptive (Proposition 5), and the associated order coincides with what is known in the literature as the partial correctness order (Corollary 2). It is noteworthy that $\mathcal{L}(\mathcal{K})$ can be used to calculate, up-to closure, the efficient frontier of a soft CSP problem in $\mathcal{K}$ (Theorem 1).

We further improved on that by showing that each finitely downward-closed set is uniquely represented by the set of its irredundant elements (Proposition 7). Building on that we stated Theorem 2: our main result establishes a precise correspondence between the efficient frontier of the original soft problem on $\mathcal{K}$ and the best level of consistency of $\mathcal{I}(\mathcal{K})$.

Related with that, note that the sum and product operator of $\mathcal{I}(\mathcal{K})$ are optimized with respect to $\mathcal{L}_f(\mathcal{K})$ because for any pair of irredundant sets S, T we have $lubs\,(S \cup T) \subseteq lubs\, S \cup lubs\, T$ and $lubs\,(S \times_F T) \subseteq lubs\, S \times lubs\, T$.

5 Examples of Application

In this section we illustrate the expressiveness of the powerset semirings for modelling multi-objective problems. It is worth noting that the partial correctness transformation on finite representations can be used to model any multi-objective problem. The only requirement is that each criteria must be expressed over a suitable semiring $\mathcal{K}$.

5.1 Multi-Objective CSP

A *multi-objective* CSP (MC-CSP) is a soft CSP problem composed by a family of p CSPs. Each criteria can be defined over the semiring $\mathcal{K}_{CSP}$. Then, a MC-CSP problem is defined over semiring $\mathcal{I}(\mathcal{K}_{CSP_1} \times \ldots \times \mathcal{K}_{CSP_p})$.

Consider a problem with two variables $\{x, y\}$, two values in each domain $\{a, b\}$, and two criteria to be satisfied. For the first criteria, the assignments $(x = a, y = a)$, $(x = b, y = a)$, and $(x = a, y = b)$ are forbidden. For the second criteria, the assignments $(x = b, y = a)$, $(x = a, y = b)$, and $(x = b, y = b)$ are forbidden. Let

$$\mathcal{K}_{2-CSP} = \langle \{f, t\} \times \{f, t\}, \bar{\vee}, \bar{\wedge}, \langle f, f\rangle, \langle t, t\rangle\rangle$$

be the cartesian product of two semirings $\mathcal{K}_{CSP}$ (one for each criteria), where f and t are short-hands for $false$ and $true$, respectively. $\bar{\vee}$ is the pairwise $\vee$ and $\bar{\wedge}$ is the pairwise $\wedge$. Then, the problem is represented as a soft CSP $P = \langle \mathcal{C}, \mathcal{X}\rangle$ over $\mathcal{K}_{2-CSP}$, where $\mathcal{C} = \{C_x, C_y, C_{xy}\}$ is defined as

$$\begin{array}{ll} C_x(a) = C_x(b) = C_y(a) = C_y(b) = \langle t, t\rangle & \\ C_{xy}(a, a) = \langle f, t\rangle & C_{xy}(b, a) = \langle f, f\rangle \\ C_{xy}(a, b) = \langle f, f\rangle & C_{xy}(b, b) = \langle t, f\rangle \end{array}$$

The solution of P is the constraint $Sol(P)$ with support $\{x, y\}$ obtained as $C_x \bar{\wedge} C_y \bar{\wedge} C_{xy}$. Since the variables of the problem are the same as the ones in the support of the constraints, there is no need to project any variable out. Moreover, since for all η, $C_x(\eta) = C_y(\eta) = \langle t, t\rangle$ and $\langle t, t\rangle$ is the unit element with respect $\bar{\wedge}$, $Sol(P) = C_{xy}$. The best level of consistency of P is $blevel(P) = \underline{\vee}_\eta\{Sol(P)(\eta)\} = \langle t, t\rangle$.

However, we want to obtain as the best level of consistency the set of semiring values representing the efficient frontier $\mathcal{E}(P) = \{\langle f, t\rangle, \langle t, f\rangle\}$. To that end, we map the problem P to a new one, by changing the semiring $\mathcal{K}_{2-CSP}$ using the partial correctness transformation on finite representations. By applying the mapping, we obtain a problem $\mathcal{I}(P) = \langle \mathcal{C}', \mathcal{X}\rangle$ over semiring $\mathcal{I}(\mathcal{K}_{2-CSP})$, with the following constraint definition

$$\begin{array}{ll} C_x(a) = C_x(b) = C_y(a) = C_y(b) = \{\langle t, t\rangle\} & \\ C_{xy}(a, a) = \{\langle f, t\rangle\} & C_{xy}(b, a) = \{\langle f, f\rangle\} \\ C_{xy}(a, b) = \{\langle f, f\rangle\} & C_{xy}(b, b) = \{\langle t, f\rangle\} \end{array}$$

The solution of $\mathcal{I}(P)$ is the same as for P. However, its best level of consistency is $blevel(\mathcal{I}(P)) = \{\langle 0, 1\rangle, \langle 1, 0\rangle\}$, which is the efficient frontier of P. The corresponding pareto-optimal solutions are $(x = a, y = a)$ and $(x = b, y = b)$.

5.2 Multi-Objective WCSP

A *multi-objective WCSP problem* (MO-WCSP) is a soft CSP problem composed by p WCSPs. Each objective function is modelled with the semiring $\mathcal{K}_{WCSP}$. Again, a MO-WCSPs is modelled over semiring $\mathcal{I}(\mathcal{K}_{WCSP_1} \times \ldots \times \mathcal{K}_{WCSP_p})$.

Consider a knapsack problem with two objects that must be either taken or left behind. It is represented by variables $\{x, y\}$ and a two-value domain $\{t, d\}$ (t and d mean *take* or *discard*, respectively). Each object has a weight if taken ($w_x = 4$ and $w_y = 2$), and a profit loss if discarded ($p_x = 1$ and $p_y = 6$). We want to minimize the global profit loss and the global weight. The task is expressed as the simultaneous minimization of two functions: the total profit loss of discarded objects, and the total weight of selected ones.

Objectives F_i are defined over a semiring $\mathcal{K}_{WCSP_i}$. Let

$$\mathcal{K}_{2-WCSP} = \langle \mathcal{R} \times \mathcal{R}, \bar{\min}, \bar{+}, \langle \infty, \infty \rangle, \langle 0, 0 \rangle \rangle$$

be the semiring resulting from $\mathcal{K}_{WCSP_1} \times \mathcal{K}_{WCSP_2}$, while $\bar{\min}$ and $\bar{+}$ are the pairwise min and $+$, respectively.

The problem is represented as a soft CSP $P = \langle \mathcal{C}, \mathcal{X} \rangle$ over $\mathcal{K}_{2-WCSP}$, where $\mathcal{C} = \{C_x, C_y\}$ defined as,

$$\begin{array}{ll} C_x(d) = \langle 0, 1 \rangle & C_y(d) = \langle 0, 6 \rangle \\ C_x(t) = \langle 4, 0 \rangle & C_y(t) = \langle 2, 0 \rangle \end{array}$$

The solution of P is the constraint $Sol(P)$ where,

$$\begin{array}{ll} Sol(P)(dd) = \langle 0, 7 \rangle & Sol(P)(dt) = \langle 2, 1 \rangle \\ Sol(P)(tt) = \langle 6, 0 \rangle & Sol(P)(td) = \langle 4, 6 \rangle \end{array}$$

The $blevel(P) = \langle 0, 0 \rangle$, while $\mathcal{E}(P) = \{\langle 0, 7 \rangle, \langle 2, 1 \rangle, \langle 6, 0 \rangle\}$. Again, if we apply the partial correctness transformation, then $blevel(\mathcal{I}(P)) = \mathcal{E}(P)$ with pareto-optimal solutions $(x = d, y = d)$, $(x = d, y = t)$, and $(x = t, y = t)$, respectively. Observe that $\langle 4, 6 \rangle$ does not belong to $blevel(\mathcal{I}(P))$ since it is redundant with respect to $\langle 2, 1 \rangle$.

6 Conclusions and further work

Problems involving the optimization of more than one objective are ubiquitous in real world domains. They are probably the most relevant optimization problems with a partially ordered objective function. So far, nobody has yet studied in depth how to use the soft CSP framework to model multi-objective problems. The only attempt is [2], where they use the least upper bound as notion of solution, which is a relaxed notion regarding pareto-optimality.

Our paper addresses exactly this issue. For the first time, we distill a semiring able to define problems such that their best level of consistency is the efficient frontier of a multi-objective problem. This formalization is important for two main reasons: we gain some understanding of the nature of multi-objective optimization problems; and we inherit some theoretical result from the soft CSP framework.

We are currently investigating the semiring $\mathcal{S}(\mathcal{K})$ resulting from *saturated* closure, i.e., whose elements are sets S that are both downward- and upward-closed. More generally, we look for suitable constructions such that the resulting absorptive semiring turns out to be a *division* semiring, if K is so. This would allow for the application of local consistency algorithm to a larger family of case studies, as shown in [1].

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