

Adaptive estimation of the conditional cumulative distribution function from current status data

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Abstract

Consider a positive random variable of interest Y depending on a covariate X , and a random observation time T independent of Y given X . Assume that the only knowledge available about Y is its current status at time T : $\delta = \mathbb{I}_{\{Y \leq T\}}$ with \mathbb{I} the indicator function. This paper presents a procedure to estimate the conditional cumulative distribution function F of Y given X from an independent identically distributed sample of (X, T, δ) .

A collection of finite-dimensional linear subsets of $L^2(\mathbb{R}^2)$ called models are built as tensor products of classical approximation spaces of $L^2(\mathbb{R})$. Then a collection of estimators of F is constructed by minimization of a regression-type contrast on each model and a data driven procedure allows to choose an estimator among the collection. We show that the selected estimator converges as fast as the best estimator in the collection up to a multiplicative constant and is minimax over anisotropic Besov balls. Finally simulation results illustrate the performance of the estimation and underline parameters that impact the estimation accuracy.

Keywords: Current status, Interval Censoring, Distribution function, Model selection, Adaptivity, Anisotropic Besov spaces, Minimax.

1. Introduction

Current status data framework refers to a situation where the observation of a positive variable of interest Y called lifetime, is restricted to the knowledge of whether or not Y exceeds a random measure time T . More precisely,

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one only observes the time T and the “current status” of the system at time T , namely $\delta = \mathbb{I}_{\{Y \leq T\}}$ equal to 1 if $Y \leq T$ and 0 otherwise. Such data arise naturally for example in infectious disease studies, when the time Y of infection is unobserved, and a test is carried out at time T . This framework is also named interval censoring, case 1, since the observation (T, δ) indicates whether Y lies in $[0, T]$ or $(T, +\infty)$. Contrary to other censoring framework, such as the right-censoring, the variable of interest is never directly measured. Nevertheless, as the intervals $[0, T]$ and $(T, +\infty)$ are different for each observation, quantities related to Y can be inferred.

The design considered in this paper is classical in current status data framework (e.g. [Cheng and Wang \(2011\)](#), [van der Vaart and van der Laan \(2006\)](#)): the survival and observation times Y and T are assumed to be independent given a uni-dimensional covariate X directly measured. In the first place, no smoothness conditions are assumed on the variables (X, Y, T) . More precisely, the following i.i.d. sample is observed

$$(X_i, T_i, \delta_i)_{i=1, \dots, n} \tag{1}$$

with the survival times $(Y_i)_{i=1, \dots, n}$ independent from the observation times $(T_i)_{i=1, \dots, n}$ conditional to the covariates $(X_i)_{i=1, \dots, n}$, and $(\delta_i)_{i=1, \dots, n}$ denoting the indicators $(\mathbb{I}_{\{Y_i \leq T_i\}})_{i=1, \dots, n}$. We aim to estimate the cumulative distribution function (c.d.f.) F of Y given a covariate X , namely

$$F(x, y) = P[Y \leq y | X = x].$$

Current status data have been widely studied over the past two decades. Most results about nonparametric estimation of the survival function are based on the NPMLE (Nonparametric Maximum Likelihood Estimator). [Groeneboom and Wellner \(1992\)](#) prove that the NPMLE is pointwise convergent with the optimal rate $n^{-1/3}$, and [van de Geer \(1993\)](#) establishes a similar result for the L^2 -risk under the assumption of continuous survival time density. This unusual rate of convergence differs from the uncensored and right-censored designs, in which the distribution function can be estimated with the parametric rate of convergence $n^{-1/2}$.

More recently, estimators developed from the NPMLE take into account the regularity of the function. [Hudgens et al. \(2007\)](#) build three estimators derived from the NPMLE, and compare their performances on simulated and real data. [van der Vaart and van der Laan \(2006\)](#) apply smoothing methods to the NPLME to estimate the survival function in presence of

high-dimensional covariates. [Birgé \(1999\)](#) proposes an easily computable histogram estimator which reaches the minimax rate of convergence. The procedures developed in these papers assume that the regularity of the target function is known.

Few papers propose estimators that automatically adapt to the unknown regularity of the function. [Ma and Kosorok \(2006\)](#) introduce a NPMLE and a least square estimator on Sobolev classes, and select the regularity parameter with a penalized criterion. [Brunel and Comte \(2009\)](#) draw a parallel between current status data and bounded regression framework through a least-square estimator combined with a model selection procedure based on a more easily computable penalty function. As far as the author knows, estimation of cumulative distribution function from current status data in presence of covariates has not been handled yet in an adaptive framework. Moreover, lower-bounds on classical regularity spaces are not available. This paper aims to address these two questions.

First of all, the procedure from [Brunel and Comte \(2009\)](#) is extended in order to include a covariate. For every $i = 1, \dots, n$,

$$\mathbb{E}[\delta_i | X_i, T_i] = \mathbb{E}[\mathbb{1}_{\{Y_i \leq T_i\}} | X_i, T_i] = \mathbb{P}[Y_i \leq T_i | X_i, T_i] = F(X_i, T_i).$$

Therefore, F is the regression function of δ_i on (X_i, T_i) and a least-square contrast is considered. A collection of linear spaces of $L^2(\mathbb{R} \times \mathbb{R}^+)$ is built by tensor product of classical approximation subspaces of $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^+)$. Finally, a model selection estimator is computed by penalization of the least-square contrast following the model selection procedure from [Birgé and Massart \(1998\)](#).

The loss function usually considered in model selection procedures is associated to the L^2 -norm. Nevertheless, the least-square contrast which quantifies the goodness of fit on the observations provides a natural control of the risk associated to the empirical norm $\|t\|_n^2 = (1/n) \sum_{i=1}^n t^2(X_i, T_i)$ which represents an intermediate result in the computation of the integrated risk in most model selection procedures (see e.g. [Massart \(2007\)](#), Comments of Theorem 8.3). In this paper, we have chosen to first display the control of the empirical risk conditional to the observations $\{(X_i, T_i)\}$ and then derive the result for the L^2 -risk under additional assumptions in order to emphasize the role played by each hypothesis. Besides, the fixed design context with non-random observations $\{(X_i, T_i)\}$ can be directly handled.

On complementary interest to the adaptivity on decomposition bases, is the study of the rate of convergence on spaces of regularity. Anisotropic Besov spaces, which account for different regularities of the target function F with respect to the covariate X and the survival time Y , are considered. On the one hand, the rate of convergence of our adaptive estimator is evaluated, exhibiting classical rates in a regression context, but unusual in c.d.f. estimation. On the other hand, the minimax rate of convergence is considered as a criterion to evaluate the fundamental limit of the current status framework. The combination of these two analyses assesses the optimality of our estimator. Results in a context without covariate can be derived as a particular case, proving the optimality of the estimators proposed by [Ma and Kosorok \(2006\)](#) and [Brunel and Comte \(2009\)](#).

Furthermore, we analyze the performance of our estimator on simulated data. In a right-censoring context, the rate of censoring defined as the proportion of censored observations is a well-identified parameter of the quality of the estimation. Since the survival time is never observed in current status data, this quantity makes no sense. Nevertheless, in our numerical study, we exhibit a parameter which impacts the estimation from current status data, and may represent the counterpart of the rate of censoring on right-censored data.

The paper is organized as follows. Section 2 introduces the notations. The estimation procedure is presented in Section 3. The upper-bounds for the empirical and integrated loss functions are presented in Section 4. The minimax study over anisotropic Besov balls is conducted in Section 5. Section 6 displays the performances of the estimator on simulated data. General comments and perspectives for further developments are presented in Section 7. The proofs are gathered in Section 8.

2. Notations

2.1. Notations

Let \mathbf{X} and \mathbf{T} denote the vectors (X_1, \dots, X_n) and (T_1, \dots, T_n) , and for every $t \in L^2(\mathbb{R} \times \mathbb{R}^+)$, let $t(\mathbf{X}, \mathbf{T}) = (t(X_1, T_1), \dots, t(X_n, T_n))$. For every $(V_i, W_i)_{i=1, \dots, n}$ i.i.d. random variables, f_V denotes the density of V_i and $f_{(V,W)}$ the density of the couple (V_i, W_i) .

Let $A = A_1 \times A_2$ be a subset of $\mathbb{R} \times \mathbb{R}^+$ with A_1 and A_2 compact intervals. For every $t, s \in L^2(A)$, we define the empirical norm and scalar product

relative to the sample $(X_i, T_i)_{i=1, \dots, n}$:

$$\langle s, t \rangle_n = \frac{1}{n} \sum_{i=1}^n s(X_i, T_i) t(X_i, T_i) \quad \text{and} \quad \|s\|_n^2 = \langle s, s \rangle_n$$

and the associated integrated norm and scalar product:

$$\langle s, t \rangle_{f(x,T)} = \int_{x \in A_1} \int_{u \in A_2} s(x, u) t(x, u) f_{(X,T)}(x, u) dx du \quad \text{and} \quad \|s\|_{f(x,T)}^2 = \langle s, s \rangle_{f(x,T)}.$$

Finally, for every finite set \mathcal{S} , $|\mathcal{S}|$ denotes the cardinal of \mathcal{S} .

3. Model selection estimator

3.1. Collection of linear models

We build a collection of linear subspaces of $L^2(A)$, referred to as *models*, by tensor product of finite-dimensional linear subspaces of $L^2(A_1)$ and $L^2(A_2)$. For $j = 1$ or 2 , let

$$\mathcal{M}_n^{(j)} = \{S_{m_j}^{(j)}, m_j \in I_n^{(j)}\} \quad (2)$$

be a collection of linear subspaces of $L^2(A_j)$ with finite dimension $D_{m_j}^{(j)}$. Let $I_n = I_n^{(1)} \times I_n^{(2)}$. For every $m = (m_1, m_2) \in I_n$, we define

$$S_m = S_{m_1}^{(1)} \otimes S_{m_2}^{(2)} = \left\{ t : A \rightarrow \mathbb{R}, \quad t(x, y) = \sum_{(k,l) \in J_m} a_{k,l} \phi_k^{m_1}(x) \psi_l^{m_2}(y) \right\} \quad (3)$$

with $(\phi_k^{m_1})_{k=1, \dots, D_{m_1}^{(1)}}$ an orthonormal basis of $S_{m_1}^{(1)}$, $(\psi_l^{m_2})_{l=1, \dots, D_{m_2}^{(2)}}$ an orthonormal basis of $S_{m_2}^{(2)}$ and

$$J_m = ((1, 1), \dots, (1, D_{m_2}^{(2)}), (2, 1), \dots, (2, D_{m_2}^{(2)}), \dots, (D_{m_1}^{(1)}, 1), \dots, (D_{m_1}^{(1)}, D_{m_2}^{(2)})) \quad (4)$$

For every $m = (m_1, m_2)$, S_m is a linear subspace of $L^2(A)$ with dimension $D_m = D_{m_1}^{(1)} D_{m_2}^{(2)}$. Then the collection of models is defined as $\mathcal{M}_n = \{S_m, m \in I_n\}$. Once fixed the collection \mathcal{M}_n , a bijection is established between the indexes $\{m \in I_n\}$ and the linear spaces $\{S_m \in \mathcal{M}_n\}$. Therefore, the term "model" will indistinctly refer either to m or S_m . We assume that $\mathcal{M}_n^{(1)}$ and

$\mathcal{M}_n^{(2)}$ satisfy the following assumption.

(H) Let $j = 1$ or 2 . For every $b > 0$, there exists a constant B_j such that

$$\sum_{m_j \in I_n^{(j)}} \exp\left(-b\sqrt{D_{m_j}^{(j)}}\right) \leq B_j, \quad \forall n \in \mathbb{N}^*.$$

3.2. Least-square estimators

For every $t \in L^2(A)$, we consider the least-square contrast

$$\gamma_n(t) = \frac{1}{n} \sum_{i=1}^n (\delta_i - t(X_i, T_i))^2. \quad (5)$$

For every model $S_m \in \mathcal{M}_n$, we denote by \widehat{F}_m the minimum contrast estimator on S_m :

$$\widehat{F}_m = \arg \min_{t \in S_m} \gamma_n(t) = \sum_{(k,l) \in J_m} \widehat{a}_{k,l} \phi_k^{m_1}(\cdot) \psi_l^{m_2}(\cdot) \quad (6)$$

Equation (6) amounts to state that the partial derivatives $\partial \gamma_n(\widehat{F}_m) / \partial a_{k,l}$ are equal to zero for every $(k, l) \in J_m$. Therefore, the column vector of coefficients $\widehat{A}_m = [\widehat{a}_{k,l}]_{(k,l) \in J_m}$ satisfies

$$\widehat{G}_m \widehat{A}_m = \widehat{V}_m \quad (7)$$

with

$$\widehat{G}_m = \left[\frac{1}{n} \sum_{i=1}^n \phi_k^{m_1}(X_i) \psi_l^{m_2}(T_i) \phi_{k'}^{m_1}(X_i) \psi_{l'}^{m_2}(T_i) \right]_{((k,l), (k',l')) \in J_m^2}$$

the $D_m \times D_m$ -square Gram matrix related to $\{\phi_k^{m_1} \psi_l^{m_2}\}_{(k,l) \in J_m}$ for the scalar product $\langle \cdot, \cdot \rangle_n$ and

$$\widehat{V}_m = \left[\frac{1}{n} \sum_{i=1}^n \delta_i \phi_k^{m_1}(X_i) \psi_l^{m_2}(T_i) \right]_{(k,l) \in J_m}$$

a column vector.

Comment on existence and unicity of \widehat{F}_m . Equation (7) has a unique solution \widehat{A}_m if and only if the matrix \widehat{G}_m is invertible. Nevertheless, we show

that the estimator \widehat{F}_m exists almost surely and is uniquely defined on the set (\mathbf{X}, \mathbf{T}) .

Consider the observed sample (1). Let \mathcal{S} be a finite dimensional linear subset of $L^2(A)$ and Φ be the linear application from \mathcal{S} to \mathbb{R}^n defined as $\Phi(t) = t(\mathbf{X}, \mathbf{T})$. We denote by \mathcal{S}^* the linear subset of \mathcal{S} such that:

$$\mathcal{S}^* \oplus \text{Ker}(\Phi) = \mathcal{S} \quad (8)$$

with $\text{Ker}(\Phi)$ the null space of Φ . \mathcal{S}^* is the largest linear subset of \mathcal{S} on which $\|\cdot\|_n$ satisfies the separation property: $\|t\|_n = 0 \Rightarrow t = 0$. The restriction Φ^* of Φ to \mathcal{S}^* draws a bijection from \mathcal{S}^* into the image $\Phi(\mathcal{S})$ of Φ . Let $Z_{\mathcal{S}}$ be the projection of $(\delta_1, \dots, \delta_n)$ on $\Phi(\mathcal{S})$. There exists a unique function $\widehat{F}_{\mathcal{S}}^* \in \mathcal{S}^*$ such that $\Phi(\widehat{F}_{\mathcal{S}}^*) = Z_{\mathcal{S}}$. Furthermore, $\widehat{F}_{\mathcal{S}}^* = \arg \min_{t \in \mathcal{S}} \gamma_n(t)$ and for any minimiser $\widehat{F}_{\mathcal{S}}$ of γ_n over \mathcal{S} , $\widehat{F}_{\mathcal{S}}^*$ and $\widehat{F}_{\mathcal{S}}$ are equal on (\mathbf{X}, \mathbf{T}) .

3.3. Model selection estimator

For every $m \in \mathcal{M}_n$, let F_m be a minimizer of $\|F - t\|_n^2$ for $t \in S_m$. The empirical risk of the least-square estimator \widehat{F}_m has the following bias-variance decomposition:

$$\mathbb{E} \left[\|\widehat{F}_m - F\|_n^2 | (\mathbf{X}, \mathbf{T}) \right] = \mathbb{E} \left[\|\widehat{F}_m - F_m\|_n^2 | (\mathbf{X}, \mathbf{T}) \right] + \|F_m - F\|_n^2.$$

The bias term $\|F_m - F\|_n^2$ is estimated by $\gamma_n(\widehat{F}_m)$.

Let S_m^* be defined by (8), \widehat{F}_m^* and F_m^* be respectively the minimisers of γ_n and $\|\cdot - F\|_n$ over S_m^* . Then $\widehat{F}_m^*(\mathbf{X}, \mathbf{T})$ and $F_m^*(\mathbf{X}, \mathbf{T})$ are respectively the projection of $(\delta_1, \dots, \delta_n)$ and $F(\mathbf{X}, \mathbf{T})$ on S_m^* for the norm $\|\cdot\|_n$. Thus, let $\{\varphi_\lambda, \lambda \in J_m^*\}$ be a $\|\cdot\|_n$ -orthonormal basis of S_m^* ,

$$\|\widehat{F}_m^* - F_m^*\|_n^2 = \|\widehat{F}_m - F_m\|_n^2 = \sum_{\lambda \in J_m^*} (\nu_n(\varphi_\lambda))^2$$

with

$$\nu_n(t) = \frac{1}{n} \sum_{i=1}^n (\delta_i - F(X_i, T_i)) t(X_i, T_i). \quad (9)$$

Moreover, the $\{\delta_i\}_{i=1, \dots, n}$ are independent conditional to (\mathbf{X}, \mathbf{T}) and $\mathbb{E}[(\delta_i - F(X_i, T_i))^2 | \mathbf{X}, \mathbf{T}] \leq 1/4$ as variance of a Bernoulli variable, which entails:

$$\mathbb{E} [(\nu_n(t))^2 | \mathbf{X}, \mathbf{T}] \leq \frac{1}{4n^2} \sum_{i=1}^n t^2(X_i, T_i) = \frac{\|t\|_n^2}{4n}, \quad \forall t \in L^2(A). \quad (10)$$

Therefore, the variance term $\mathbb{E} \left[\|\widehat{F}_m - F_m\|_n^2 | (\mathbf{X}, \mathbf{T}) \right]$ is upper bounded by $(1/4)D_m/n$ and a model \widehat{m} which minimizes the estimated bias-variance sum is selected:

$$\widehat{m} = \arg \min_{m \in I_m} \left[\gamma_n(\widehat{F}_m) + \text{pen}(m) \right]$$

with $\text{pen}(m) = (\theta/4)D_m/n$ for some numerical constant $\theta > 1$.

Besides, the cumulative distribution function F lies in $[0, 1]$. We use this information to improve the estimation by constraining the values of our estimator to remain in this interval. More precisely we consider the estimator $\widetilde{F}_{\widehat{m}}$ with

$$\widetilde{F}_m(x, u) = \begin{cases} 0 & \text{if } \widehat{F}_m(x, u) < 0 \\ 1 & \text{if } \widehat{F}_m(x, u) > 1 \\ \widehat{F}_m(x, u) & \text{otherwise.} \end{cases} \quad (11)$$

Comments

- 1) The restriction imposed by (11) brings an improvement in terms of risk. Indeed, for every $(x, u) \in A$,

$$|\widetilde{F}_m(x, u) - F(x, u)| \leq |\widehat{F}_m(x, u) - F(x, u)|, \quad \forall m \in \mathcal{M}_n$$

almost surely. In particular, $\|\widetilde{F}_m - F\|_n^2 \leq \|\widehat{F}_m - F\|_n^2$. Thus, any upper-bound of $\mathbb{E}[\|\widehat{F}_m - F\|_n^2 | \mathbf{X}, \mathbf{T}]$ is also an upper-bound of $\mathbb{E}[\|\widetilde{F}_m - F\|_n^2 | \mathbf{X}, \mathbf{T}]$.

- 2) The convergence results presented in this paper are valid for any $\theta > 1$, but in practical implementation a value has to be fixed. This issue arises in most model selection procedure. It can be either calibrated on simulated data from a large number of examples, determined through a cross-validation procedure, or chosen a priori. In particular, [Massart \(2008\)](#) proposes an heuristic which validates the value $\theta = 2$ in a general context.
- 3) As F is bounded by 1, the constant involved in the penalty is purely numerical contrary to many frameworks where the penalty includes unknown parameters which have to be estimated.

4. Non-asymptotic upper-bounds

4.1. Oracle inequality for the empirical risk

The minimization of the least-square contrast (5) determines the function which offers the best fit on the sample (\mathbf{X}, \mathbf{T}) . Therefore, the empirical loss function $\|\tilde{F}_{\hat{m}} - F\|_n^2$ which measures the error of estimation on the set (\mathbf{X}, \mathbf{T}) appears as the natural risk for the estimator $\tilde{F}_{\hat{m}}$.

Theorem 4.1. *Assume that (H) holds. There exist numerical constants C_1 and C_2 such that*

$$\mathbb{E} \left[\|\tilde{F}_{\hat{m}} - F\|_n^2 | \mathbf{X}, \mathbf{T} \right] \leq C_1 \inf_{m \in I_n} \left\{ \inf_{t \in S_m} \|F - t\|_n^2 + \text{pen}(m) \right\} + \frac{C_2}{n} \quad a.s. \quad (12)$$

Comments on Theorem 4.1

- 1) Theorem 4.1 indicates that the model selection estimator realizes the trade-off between the bias and the penalty, up to a multiplicative constant. Nevertheless, since the penalty has only been stated as an upper-bound of the variance, we did not strictly prove that the sum $(\inf_{t \in S_m} \|F - t\|_n^2 + \text{pen}(m))$ has the order of the bias-variance sum. The minimax study conducted in Section 5 is necessary to ensure that the penalty has the order of the variance, and therefore assess the adaptivity of the estimator.
- 2) The result of Theorem 4.1 holds for any value of the sample (\mathbf{X}, \mathbf{T}) , regardless of its distribution. In particular, it handles the situation where the observation times \mathbf{T} and the covariates \mathbf{X} are fixed by the experimental design.

4.2. Oracle inequality for the L^2 -risk

Theorem 4.1 states the adaptivity of the model selection estimator for the empirical risk which naturally arises from the least-square contrast. Nevertheless, the loss function considered in Theorem 4.1 is data dependent, and the control of the classical L^2 -risk provides more interpretable results. In this section, we derive an oracle inequality for the L^2 -risk under the following assumptions.

(**A₁**) For $i = 1, \dots, n$, (X_i, T_i) has a density $f_{(X,T)}$ bounded by $h_0 > 0$ and $h_1 < +\infty$ on set A :

$$h_0 \leq f_{(X,T)}(x, u) \leq h_1, \quad \forall (x, u) \in A.$$

(**A₂**) For $j = 1$ and 2 ,

$$|\mathcal{M}_n^{(j)}| \leq P^{(j)}(n), \quad \forall n \in \mathbb{N} \tag{13}$$

for some polynomial $P^{(j)}$. Moreover, there exists a model $S_n^{(j)} \in \mathcal{M}_n^{(j)}$ of dimension $N_n^{(j)}$ such that, for every $m_j \in I_n^{(j)}$, $S_{m_j}^{(j)} \subset S_n^{(j)}$ and $N_n^{(1)} N_n^{(2)} \leq \sqrt{n}/\log n$.

(**A₃**) There exists a positive constant K_1 (resp. K_2) such that, for every $m_1 \in I_n^{(1)}$ (resp. $m_2 \in I_n^{(2)}$),

$$\sup_{x \in A_1} \sum_{k=1}^{D_{m_1}^{(1)}} (\phi_k^{m_1}(x))^2 \leq K_1 D_{m_1}^{(1)} \quad \left(\text{resp.} \quad \sup_{u \in A_2} \sum_{l=1}^{D_{m_2}^{(2)}} (\psi_l^{m_2}(u))^2 \leq K_2 D_{m_2}^{(2)} \right).$$

Assumption (**A₂**) refers to the number of models in the collection, whereas (**A₃**) is related to the nature of the models.

Examples of models. Assumption (**A₃**) holds when the collections $\mathcal{M}_n^{(1)}$ and $\mathcal{M}_n^{(2)}$ consist of the following models.

- (i) $S_{m_j}^{(j)}$ is the set of piecewise polynomials with maximum degree s_j and step $\lg(A_j)/D_{m_j}^{(j)}$ where $\lg(A_j)$ denotes the length of A_j .
- (ii) $S_{m_j}^{(j)} = \text{vect}\{\chi_{l,k}, l \leq m_j, k \in \mathbb{Z}\}$ where χ is a bounded mother wavelet with regularity s_j , $\chi_{l,k}(x) = 2^{l/2} \chi(2^l x - k)$ and $D_{m_j}^{(j)} = 2^{m_j}$.
- (iii) $S_{m_j}^{(j)}$ is the set of trigonometric polynomials with maximum degree $D_{m_j}^{(j)}$.

Theorem 4.1 leads to the following result.

Theorem 4.2. *Assume that (\mathbf{H}) , (\mathbf{A}_1) , (\mathbf{A}_2) and (\mathbf{A}_3) hold. Then*

$$\mathbb{E} \left[\|\tilde{F}_{\hat{m}} - F\|^2 \right] \leq C_3 \inf_{m \in I_n} \left\{ \inf_{t \in S_m} \|F - t\|^2 + \text{pen}(m) \right\} + \frac{C_4}{n} \quad (14)$$

for some constant (C_3, C_4) depending on (h_0, h_1, K) .

5. Minimax study on Besov balls

Theorem 4.2 states the adaptivity of the model selection estimator i.e. its optimality among the collection of estimators. The minimax rate of convergence, defined as the rate of the best estimator for the worst function among a given class of regularity, assesses optimality in a more general sense. In this section, we prove that the model selection estimator is minimax over anisotropic Besov balls.

5.1. Minimax rate of convergence

Let \mathcal{F} be a set of conditional cumulative distribution functions on A and $(r_n)_{n \in \mathbb{N}}$ a sequence of positive numbers. The sequence $(r_n)_{n \in \mathbb{N}}$ is the minimax rate of convergence of F over \mathcal{F} , defined up to a constant, if there exist two constants c and C such that

$$c \leq \inf_{\hat{F}_n} \sup_{F \in \mathcal{F}} \left(r_n^{-1} \mathbb{E}[\|\hat{F}_n - F\|^2] \right) \leq C$$

with the infimum taken over all possible estimators \hat{F}_n . The minimax rate corresponds to the rate of convergence of the best possible estimator of the function F , based on the sample $(X_i, T_i, \delta_i)_{i=1, \dots, n}$.

5.2. Definition of anisotropic Besov balls

We recall the definition of two-dimensional anisotropic Besov spaces stated for example in Hochmuth (2002). Let $f \in L^2(A)$. For $j = 1$ or 2 , $r \in \mathbb{N}^*$ and $h > 0$, let

$$\Delta_{h,j}^r(f)(x, y) = \sum_{k=0}^r \binom{r}{k} (-1)^{r-k} f((x, y) + k h e_j)$$

be the directional partial difference operator for every $(x, y) \in A_{h,j}^r$ where $A_{h,j}^r = \{(x, y) \in A, (x, y) + r h e_j \in A\}$ and (e_1, e_2) is the canonical basis of \mathbb{R}^2 . For $t > 0$, let $A_{r,j}(f, t, A) = \sup_{|h| \leq t} \|\Delta_{h,j}^r(f)(x, y)\|_{L^2(A_{h,j}^r)}$ be the directional

modulus of smoothness for the L^2 -norm. Let $\beta = (\beta_1, \beta_2) \in (\mathbb{R}_+^*)^2$ and $r_j = \lfloor \beta_j \rfloor + 1$. We define the anisotropic Besov space of parameters $(\beta, 2, \infty)$ as

$$\mathcal{B}_{2,\infty}^\beta(A) = \left\{ f \in L^2(A), |f|_{\mathcal{B}_{2,\infty}^\beta(A)} < +\infty \right\}$$

with

$$|f|_{\mathcal{B}_{2,\infty}^\beta(A)} = \sup_{t>0} \left[t^{-\beta_1} A_{r_1,1}(f, t, A) + t^{-\beta_2} A_{r_2,2}(f, t, A) \right].$$

The Besov norm on $\mathcal{B}_{2,\infty}^\beta(A)$ is defined by $\|f\|_{\mathcal{B}_{2,\infty}^\beta(A)} = |f|_{\mathcal{B}_{2,\infty}^\beta(A)} + \|f\|$ and for $L > 0$, we consider the anisotropic Besov ball:

$$\mathcal{B}_{2,\infty}^\beta(A, L) = \left\{ f \in \mathcal{B}_{2,\infty}^\beta(A), \|f\|_{\mathcal{B}_{2,\infty}^\beta(A)} \leq L \right\}.$$

5.3. Upper bound of the L^2 -risk on anisotropic Besov balls

The following corollary of Theorem 4.2 assesses the rate of convergence of the model selection estimator on anisotropic Besov balls.

Corollary 5.1. *Assume that $F \in \mathcal{B}_{2,\infty}^\beta(A, L)$ for some $L > 0$ and $\beta = (\beta_1, \beta_2) \in (\mathbb{R}_+^*)^2$ and \mathcal{M}_n is set up from models (i), (ii) with $s_j > \beta_j - 1$, or (iii). Then*

$$\inf_{t \in \mathcal{S}_m} \|F - t\| \leq C_0 \left((D_{m_1}^{(1)})^{-\beta_1} + (D_{m_2}^{(2)})^{-\beta_2} \right). \quad (15)$$

for some positive constant C_0 . Moreover assume that (\mathbf{A}_1) and (\mathbf{A}_2) hold, and

$$N_n^{(j)} \leq \left(\frac{n}{\log^2 n} \right)^{1/4} \quad \text{for } j = 1, 2. \quad (16)$$

Then

$$\mathbb{E} \left[\|\tilde{F}_{\hat{m}} - F\|^2 \right] \leq C_5 n^{-\bar{\beta}/(\bar{\beta}+1)}$$

for some positive constant C_5 , with $\bar{\beta} = 2\beta_1\beta_2/(\beta_1 + \beta_2)$ the harmonic mean of (β_1, β_2) .

The upper-bound of the bias term (15) is computed in Lacour (2007) based on results from Hochmuth (2002) and Nikol'skii (1975). Plugging this

result into Theorem 4.2 provides the rate of convergence of the bias-variance sum:

$$\inf_{t \in S_m} \|F - t\|^2 + \text{pen}(m) \leq C_6 \left\{ \left((D_{m_1}^{(1)})^{-2\beta_1} + (D_{m_2}^{(2)})^{-2\beta_2} \right) + \frac{D_{m_1}^{(1)} D_{m_2}^{(2)}}{n} \right\} \quad (17)$$

and the trade-off between the two terms in the right hand of (17) is achieved for a model (m_1, m_2) satisfying:

$$D_{m_1}^{(1)} \propto n^{\beta_2/(\beta_1+\beta_2+2\beta_1\beta_2)} \quad \text{and} \quad D_{m_2}^{(2)} \propto n^{\beta_1/(\beta_1+\beta_2+2\beta_1\beta_2)},$$

with \propto denoting proportionality. Moreover, Assumption (16) guarantees the existence of a model (\bar{m}_1, \bar{m}_2) such that

$$1 \leq D_{\bar{m}_1}^{(1)} n^{-\beta_2/(\beta_1+\beta_2+2\beta_1\beta_2)} \leq 2 \quad \text{and} \quad 1 \leq D_{\bar{m}_2}^{(2)} n^{-\beta_1/(\beta_1+\beta_2+2\beta_1\beta_2)} \leq 2$$

for $\beta_1, \beta_2 > 1$. Therefore,

$$\left((D_{\bar{m}_1}^{(1)})^{-2\beta_1} + (D_{\bar{m}_2}^{(2)})^{-2\beta_2} \right) + \frac{D_{\bar{m}_1}^{(1)} D_{\bar{m}_2}^{(2)}}{n} \propto C_7 n^{-\bar{\beta}/(\bar{\beta}+1)},$$

which concludes the proof of Corollary 5.1. \square

Comments. Condition $\beta_1, \beta_2 > 1$ in Corollary 5.1 can be extended to

$$(\beta_1, \beta_2) \in (\beta_1^*, +\infty) \times (\beta_2^*, +\infty)$$

for a known couple (β_1^*, β_2^*) which satisfies $\bar{\beta}^* \geq 1$, with $\bar{\beta}^*$ the harmonic mean of β_1^* and β_2^* , by considering $N_n^{(1)}$ and $N_n^{(2)}$ such that

$$N_n^{(1)} \leq (\log n)^{-1/2} n^{\beta_1^*/(\beta_1^*+\beta_2^*+2\beta_1^*\beta_2^*)} \quad \text{and} \quad N_n^{(2)} \leq (\log n)^{-1/2} n^{\beta_2^*/(\beta_1^*+\beta_2^*+2\beta_1^*\beta_2^*)}.$$

This alternative assumption takes into account a priori knowledge on the regularity of F , through an appropriate choice of $(N_n^{(1)}, N_n^{(2)})$. Thus, the estimation would be optimized by considering a smaller maximum size of models $(N_n^{(j)})$ in the direction where F is more regular.

5.4. Lower bound of the minimax rate of convergence

Theorem 5.1. *Let $\beta = (\beta_1, \beta_2) \in (0, +\infty) \times (1, +\infty)$. Assume that $h_1 = \|f_{(X,T)}\|_\infty < +\infty$, then there exists a constant c which depends on (β, L, h_1) such that*

$$\inf_{\hat{F}_n} \sup_{F \in \mathcal{B}_{2,\infty}^\beta(A,L)} \mathbb{E} \left[n^{-\bar{\beta}/(\bar{\beta}+1)} \|\hat{F}_n - F\|^2 \right] \geq c.$$

with the infimum taken over all possible estimators \hat{F}_n based on sample (1)

5.5. Optimality of the model selection estimator in the minimax sense

Let $\beta_1, \beta_2 > 1$. By combining the results from Theorem 4.2 and 5.1, we obtain:

$$c \leq \inf_{\hat{F}_n} \sup_{F \in \mathcal{B}_{2,\infty}^\beta(A,L)} \mathbb{E} \left[n^{-\bar{\beta}/(\bar{\beta}+1)} \|\hat{F}_n - F\|^2 \right] \leq \sup_{F \in \mathcal{B}_{2,\infty}^\beta(A,L)} \mathbb{E} \left[n^{-\bar{\beta}/(\bar{\beta}+1)} \|\hat{F}_{\hat{m}} - F\|^2 \right] \leq C$$

for some positive constants c and C . Therefore, the minimax rate of convergence over $\mathcal{B}_{2,\infty}^\beta(A,L)$ is $n^{-\bar{\beta}/(\bar{\beta}+1)}$, and the model selection estimator $\hat{F}_{\hat{m}}$ is optimal in the minimax sense.

Comments.

1. In the context where Y and T do not depend on a covariate, the minimax rate of convergence on the Besov balls of regularity β is $n^{-2\beta/(2\beta+1)}$. Heuristically, this situation corresponds to an "infinite" regularity β_1 with respect to the first variable, the harmonic mean $\bar{\beta}$ being equal to $2\beta_2$. The formal proof for the lower bound is given in Section 8, and the upper bound is provided by the estimator from Brunel and Comte (2009).
2. The rate $n^{-\bar{\beta}/(\bar{\beta}+1)}$ corresponds to the minimax rate in multivariate regression estimation (e.g. Gaïffas and Lecué (2011)) and differs from the rate $n^{-2\beta_1/(2\beta_1+1)}$ obtained on right-censored data, with β_1 the c.d.f. with respect to the covariate X (Brunel et al. (2007)). This result is coherent with the results from Brunel and Comte (2009) for independent survival and observation times: whereas the c.d.f. is estimated at the nonparametric rate $1/n$ from right-censored data, in the interval censoring framework the minimax rate over Besov balls is the classical regression estimation rate $n^{2\alpha/(2\alpha+1)}$ with α the regularity of the c.d.f.

6. Numerical study

6.1. Numerical implementation

We implement the model selection estimator of the conditional c.d.f. on a collection of models generated by histogram bases. The constant θ in the penalty is set to 2 referring to Massart (2008). Moreover, our studies indicate that the results are robust with respect to the value of the constant (not shown).

6.2. Two examples: multiplicative and additive effects of the variable X

Model 1: additive effect of X . We consider the following distribution of (X, Y, T) :

$$(\text{Mod 1}) \begin{cases} X \sim \Gamma(k = 1, \theta = 1) \\ Y = X + \varepsilon \quad \text{with} \quad \varepsilon \sim \Gamma(k = 3, \theta = 2) \\ T = X + \varepsilon' \quad \text{with} \quad \varepsilon' \sim \Gamma(k = 3, \theta = 2) \end{cases}$$

where $\Gamma(k, \theta)$ denotes the gamma distribution with shape k and scale θ . The conditional cumulative distribution function $F(x, y) = \phi(y - x, k = 3, \theta = 2)$, where ϕ denotes the density of the gamma distribution, is estimated on the set $A = [0.1, 3] \times [3.1, 19]$. The density $f_{(X,T)}$ is lower bounded by $h_0 = 3.10^{-5}$ on A .

Model 2: multiplicative effect of X . We consider the following distribution of (X, Y, T) :

$$(\text{Mod 2}) \begin{cases} X \sim \Gamma(k = 1.5, \theta = 2) \\ Y = X \times \varepsilon \quad \text{with} \quad \varepsilon \sim \Gamma(k = 3, \theta = 1) \\ T = X \times \varepsilon' \quad \text{with} \quad \varepsilon' \sim \Gamma(k = 3, \theta = 1) \end{cases}$$

The conditional cumulative distribution function $F(x, y) = \phi(y/x, k = 3, \theta = 1)$ is estimated on the set $A = [1, 10] \times [1, 20]$. The density $f_{(X,T)}$ is lower bounded by $h_0 = 10^{-7}$ on A .

For models 1 and 2, samples $(X_i, Y_i, T_i)_{i=1, \dots, n}$ of size $n = (500, 1000, 5000)$ are generated, and the model selection estimators $\tilde{F}_{\hat{m}}$ is computed from the sample $(X_i, T_i, \delta_i = \mathbb{1}_{\{Y_i \leq T_i\}})_{i=1, \dots, n}$. The results are displayed in Figure 1. Each row corresponds to a simulation model. The first column presents the target function F and columns 2 to 4 present the estimators for sample size $n = (500, 1000, 5000)$. The same functions are plotted in Figures 2 and 3 for a fixed x and y respectively. As expected, the estimation gets better when the sample size increases, but we notice that a large sample size is required in order to get correct estimation. Indeed, on the one hand, the nature of the current status data provides less precise information about the variable of interest than uncensored or right-censored data, and on the other hand, the transition from uni- to bi-dimensional framework induces a loss in estimation quality for a given sample size.

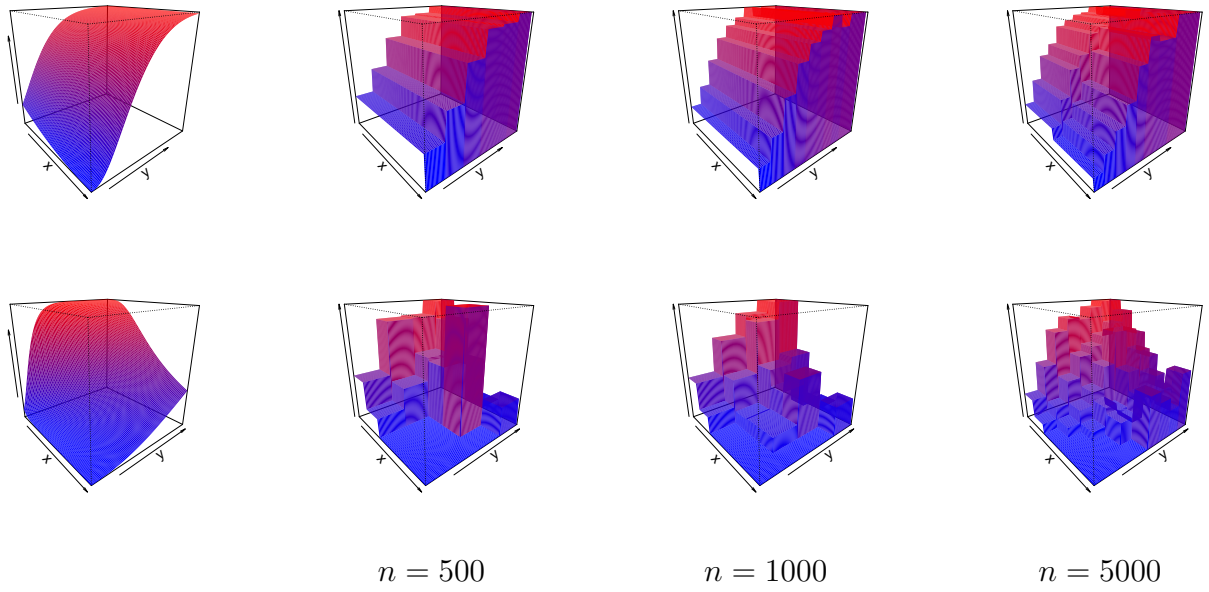


Figure 1: Conditional cumulative distribution function F (first column) and model selection estimator $\hat{F}_{\hat{m}}$ for sample size $n = (500, 1000, 5000)$. First row: model 1; Second row: model 2.

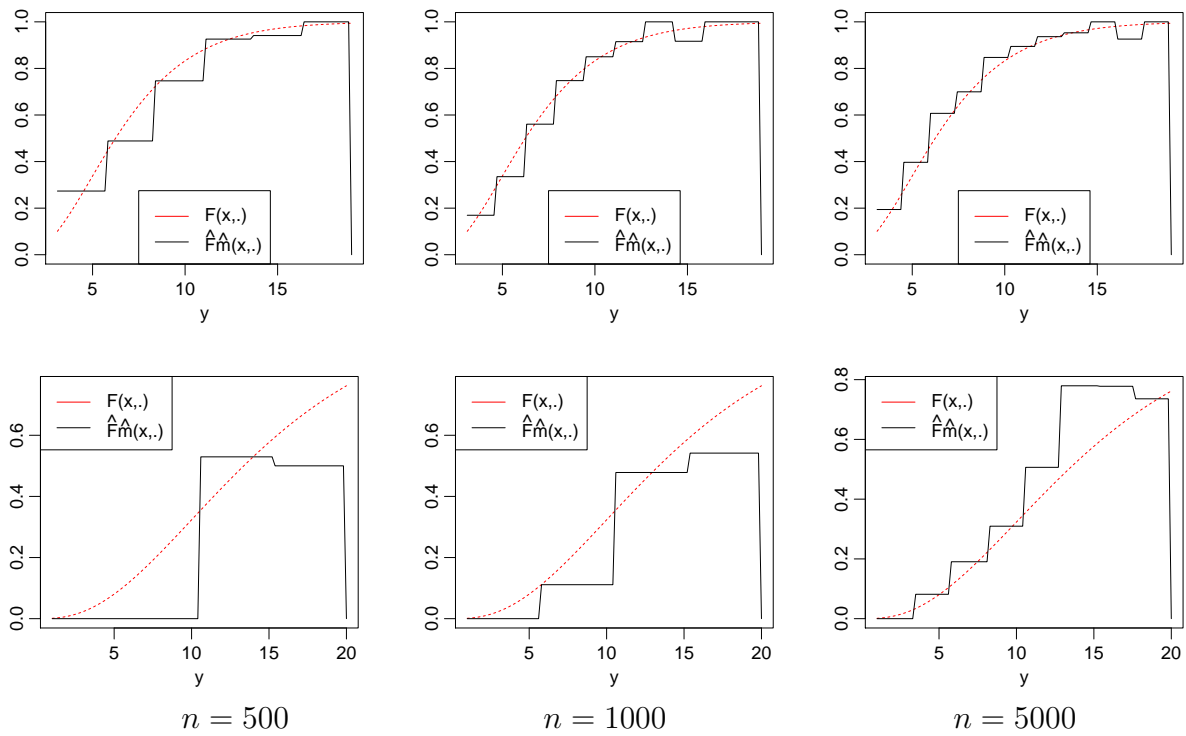


Figure 2: Model selection estimator $\hat{F}_{\hat{m}}$ (black solid line) and target function F (red dotted line) for a fixed value of x . First row: model 1, $x = 0.9$; second row: model 2, $x = 5$.

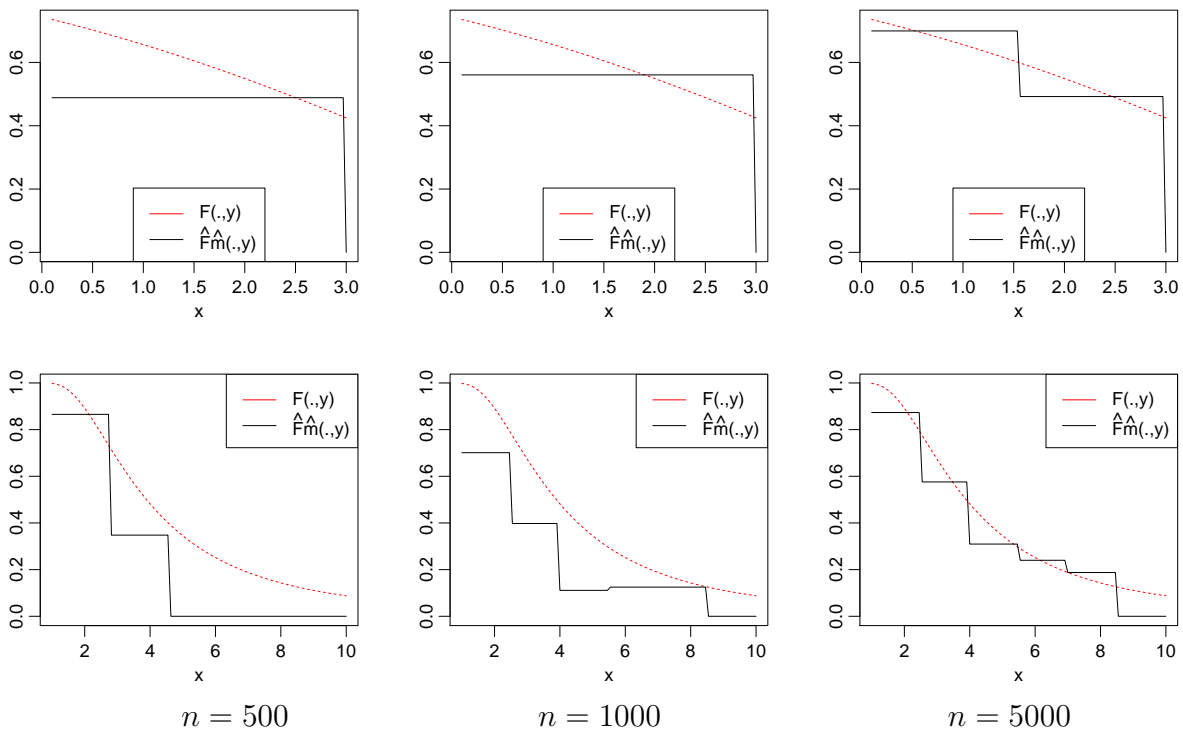


Figure 3: Model selection estimator \hat{F}_m (black solid line) and target function F (red dotted line) for a fixed value of y . First row: model 1, $y = 7.5$; Second row: model 2, $y = 10.5$.

6.3. Impact of the distance between the densities of Y and T

In a right-censoring framework, the rate of censoring, defined as the expected proportion of observations that are censored, is a well-known factor which impacts the quality of estimation: higher rate of censoring leads to poorer estimation. Indeed as the rate of censoring decreases, the proportion of survival times actually observed increases and the estimation of the survival time distribution gets better. In the current status data framework, the rate of censoring can not be defined since the survival time is never directly observed. In alternative, the distance between the densities of Y and T appears as a relevant parameter that impacts the quality of estimation. Let us heuristically explain this idea.

First of all, the estimation of F is more accurate if the observation times are concentrated in the areas where F shows the highest variations. Besides, for a given value of the variable X , the highest variations of the c.d.f. F correspond to the largest values of the conditional density $f_{Y|X}$. Therefore, the estimation of F improves as the densities of the observation time and survival time get closer, which may be quantified by the distance between the marginal densities f_Y and f_T . As a particular case, consider the situation when the supports of f_Y and f_T are disconnected: δ_i has the same value, either 0 or 1, for all i ; thus the observation set $(X_i, T_i, \delta_i)_{i=1, \dots, n}$ do not provide information about the distribution of Y . We enhance this heuristic on numerical examples.

Model 2b. We consider the same distribution as Model 2 except that an offset a is added to Y :

$$(\text{Mod 2b}) \quad \begin{cases} X \sim \Gamma(k = 1.5, \theta = 2) \\ Y = a + X \times \varepsilon \quad \text{with} \quad \varepsilon \sim \Gamma(k = 3, \theta = 1) \\ T = X \times \varepsilon' \quad \text{with} \quad \varepsilon' \sim \Gamma(k = 3, \theta = 1) \end{cases}$$

The conditional cumulative distribution function $F(x, y) = \phi(y/x, k = 3, \theta = 1)$ is estimated on the set $A = [1, 10] \times [1, 20]$. The density $f_{(X,T)}$ is lower bounded by $h_0 = 10^{-7}$ on A . We consider the $L1$ -distance between the densities of Y and T :

a	0	2	5	10
$\text{dist}(a)$	0	0.63	1.12	1.54

Table 1: L^1 -distance between f_Y and f_T for an offset a .

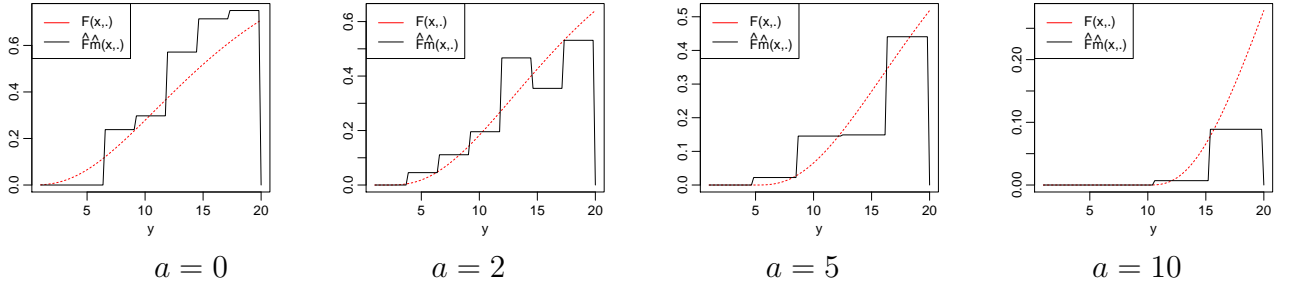


Figure 4: Results on simulation from Model 2b with offset $a = (0, 2, 5, 10)$; model selection estimator (black solid line) and target function F (red dotted line) for a fixed value of x , $x = 5.5$.

$$\begin{aligned}
 \text{dist}(a) &= \|f_Y - f_T\|_{L^1} = \int_{x=0}^{+\infty} \int_{u=0}^{+\infty} |f_{Y|X}(x, u) - f_{T|X}(x, u)| f_X(x) dx du \\
 &= \int_{x=0}^{+\infty} \int_{u=a}^{+\infty} \left| \phi\left(\frac{u-a}{x}, 3, 1\right) - \phi\left(\frac{u}{x}, 3, 1\right) \right| \phi(x, 1.5, 2) dx du
 \end{aligned}$$

The values of $\text{dist}(a)$ for $a = (0, 2, 5, 10)$ are displayed in Table 1. For each value of a , a sample $(X_i, Y_i, T_i)_{i=1, \dots, n}$ of size $n = 3000$ is generated from Model 2b, and the model selection estimator is computed on the set $A = [1, 10] \times [1, 20]$. The results are presented in Figure 4. A degradation in the estimation is observed as the distance between the densities of Y and T increases. In particular, for $a = 10$, $\text{dist}(a)$ gets close to 2, which indicates that the supports of the distribution of Y and T hardly overlap, and the estimation is very poor despite a large sample size.

6.4. Increasing rearrangement

The results displayed on Figure 2 enhance that the estimate of F is not necessarily increasing with respect to the second variable. Indeed the mean-square estimation does not account for the monotonicity of the c.d.f. The

quality of estimation can be improved by monotoning the model selection estimator $\tilde{F}_{\hat{m}}$. For an univariate function h defined on $[0, 1]$, [Chernozhukov et al. \(2007\)](#) propose a rearrangement based on the following observation: if h was non-decreasing, the quantile function of the random variable $h(U)$ with U uniformly distributed on $[0, 1]$ would be equal to h . Therefore, the rearrangement h^* of h is defined as the quantile function of $h(U)$:

$$h^*(y) = \inf \left\{ z \in \mathbb{R}, \left(\int_0^1 \mathbb{1}_{\{h(u) \leq z\}} du \right) \geq y \right\}.$$

For each $x \in A_1$, we implement this procedure to the marginal function $y \rightarrow \tilde{F}_{\hat{m}}(x, y)$ up to an affine substitution of variable which transform $A_2 = [a, b]$ into $[0, 1]$. Namely,

$$\tilde{F}_{\hat{m}}^*(x, y) = \inf \left\{ z \in \mathbb{R}, \left(\int_a^b \mathbb{1}_{\{\tilde{F}_{\hat{m}}(x, u) \leq z\}} du \right) \geq y - a \right\}, \quad \forall (x, y) \in A. \quad (18)$$

In the numerical implementation, the infimum in (18) is taken over the range of values of $\tilde{F}_{\hat{m}}(x, \cdot)$. Moreover, as $\tilde{F}_{\hat{m}}$ is piecewise constant, the rearrangement is identical on each interval and only a finite number of rearrangement has to be computed. The results are presented on [Figure 5](#).

By construction the function $\tilde{F}_{\hat{m}}^*$ is increasing with respect to the second variable. Moreover, according to Proposition 1 in [Chernozhukov et al. \(2007\)](#), the rearranged estimate presents a smaller L^2 -error than the initial estimate: for every $x \in A_1$,

$$\int_{A_2} \left(\tilde{F}_{\hat{m}}^*(x, y) - F(x, y) \right)^2 dy \leq \int_{A_2} \left(\tilde{F}_{\hat{m}}(x, y) - F(x, y) \right)^2 dy \quad a.s. \quad (19)$$

Thus the rearrangement brings an improvement in terms of L^2 -risk.

7. Discussion

7.1. Summary of the results

In the interval censoring framework, case 1, [Brunel and Comte \(2009\)](#) observe that the c.d.f. is equal to the regression function of the observation times over the indicators and propose an estimator of the c.d.f. computed by minimization of a least-square contrast on a collection of linear models

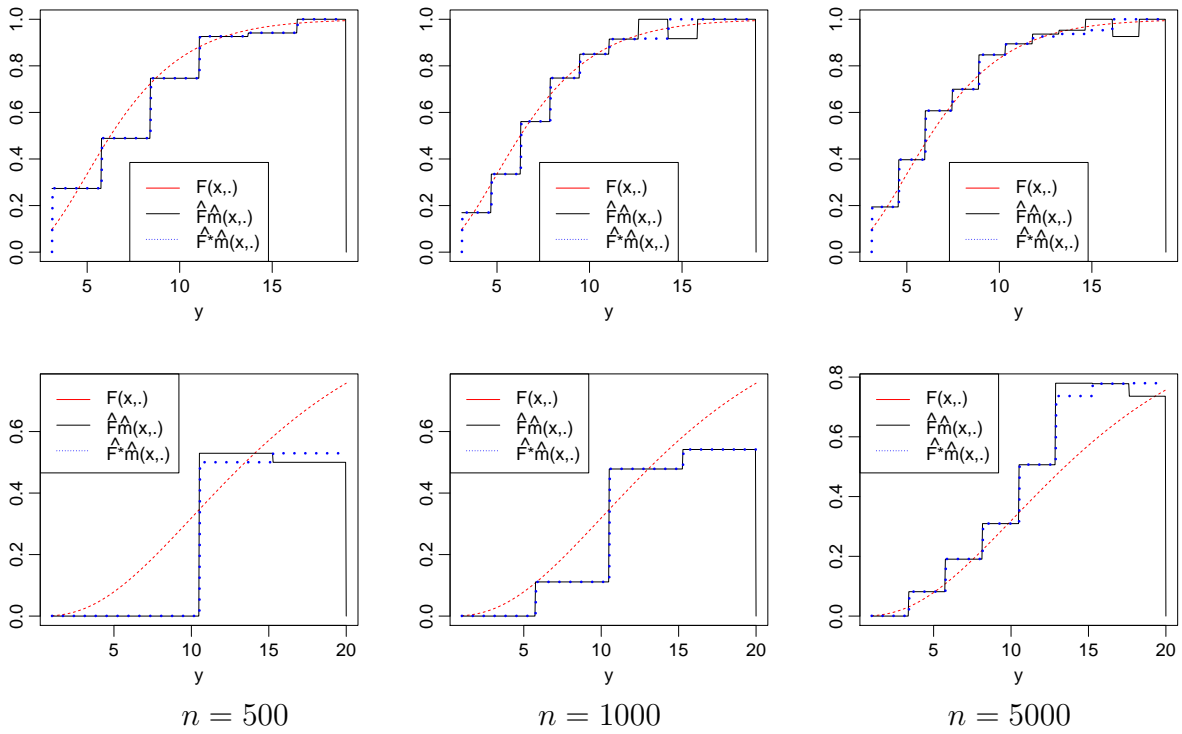


Figure 5: Model selection estimator $\hat{F}_{\hat{m}}$ (black solid line), rearranged estimator $\hat{F}_{\hat{m}^*}^*$ (blue dotted line) and target function F (red dashed line) for a fixed value of x . First row: model 1, $x = 0.9$; second row: model 2, $x = 5$.

followed by a model selection procedure. In this paper, we extend this approach to conditional c.d.f. via a collection of two-dimensional linear models built as tensor products of uni-dimensional ones. The inclusion of a covariate does not give rise to specific technical difficulties, but in order to emphasize the role played by each hypothesis, we adopt a different presentation of the results which leads to changes in the proofs.

First of all, an oracle inequality for the empirical risk conditional to the observations (\mathbf{X}, \mathbf{T}) is stated. This result only requires a limitation of the number of models in the collection with a given dimension and is valid regardless of the regularity of the variables. Besides, it directly handles the fix-designed context with non-random observation times and covariates. The main difference in the proof is the use of a concentration inequality for empirical process with independent but non identically distributed variables. An oracle inequality for the L^2 -risk on a restricted set $A \in \mathbb{R} \times \mathbb{R}^+$ is then derived under additional assumptions. First of all, the density $f_{(X,T)}$ is assumed to be bounded on A in order to guarantee the equivalence between the empirical and L^2 -norms; this condition is inherent to the design since the c.d.f can only be estimated on a set with sufficiently concentrated observations. Moreover, the construction of our collection of models by tensor product requires the set A to be a product of intervals.

The oracle inequalities state the adaptivity of the model selection estimator over the collection of models. In order to demonstrate the optimality in a more general sense, a minimax study is conducted over anisotropic Besov balls. The rate is characteristic of regression estimation but differs from the rate obtained on right-censored data.

A numerical study on simulated data with additive and multiplicative effects of the covariate is presented. We observe that a large sample size is necessary in order to get acceptable quality of estimation, due on the one hand to the multi-dimensional context, and on the other hand to the current status data framework in which poor information is brought by the censoring indicator. Besides, in alternative to the censoring rate usually considered in right-censoring, we emphasize the impact of the distance between the densities of the survival and observation times on the quality of estimation: closer densities lead to a better estimation.

7.2. Extension to multi-dimensional covariates

From a theoretical point of view, the procedure presented in this paper could be extended to covariates $X \in \mathbb{R}^k$ with $k \geq 2$ by building a collection

of models from multiple tensor products. For an index $m = (m_1, \dots, m_{k+1})$, the following model would be considered:

$$S_m = S_{m_1}^{(1)} \otimes \dots \otimes S_{m_{k+1}}^{(k+1)}$$

with $S_{m_1}^{(1)}, \dots, S_{m_k}^{(k)}$ linear subspaces of $L^2(\mathbb{R})$ and $S_{m_{k+1}}^{(k+1)}$ linear subspace of $L^2(\mathbb{R}^+)$.

Nevertheless the simulation study with a uni-dimensional covariate emphasizes the necessity of a large sample, and additional covariates could only be considered in situations with a large sample available. Besides the control of the integrated risk requires restrictions on the maximum dimension of the models. Let $N_n^{(\ell)}$ be the maximum dimension of $S_{m_\ell}^{(\ell)}$ for $\ell = 1, \dots, k+1$, assumption **(A₂)** turns into:

$$N_n^{(1)} \times \dots \times N_n^{(k+1)} \leq \sqrt{\frac{n}{\log n}}. \quad (20)$$

which is satisfied as soon as:

$$N_n^{(\ell)} \leq \left(\frac{n}{\log n} \right)^{1/2(k+1)}, \quad \forall \ell = 1, \dots, k+1.$$

Assume that F is in $\mathcal{B}_{2,\infty}^\beta(A, L)$ the anisotropic Besov ball of regularity $\beta = (\beta_1, \dots, \beta_{k+1})$ on $L^2(\mathbb{R}^k \times \mathbb{R}^+)$. Under condition (20), similarly to (17), the model selection estimator satisfies the following inequality.

$$\begin{aligned} & \mathbb{E} \left[\|\tilde{F}_{\hat{m}} - F\|^2 \right] \\ & \leq C \left\{ (D_{m_1}^{(1)})^{-2\beta_1} + \dots + (D_{m_{k+1}}^{(k+1)})^{-2\beta_{k+1}} + \frac{D_{m_1}^{(1)} \dots D_{m_{k+1}}^{(k+1)}}{n} \right\} + \frac{C'}{n} \end{aligned} \quad (21)$$

with $D_{m_\ell}^{(\ell)}$ the dimension of $S_{m_\ell}^{(\ell)}$. Thus, the model $\bar{m} = (\bar{m}_1, \dots, \bar{m}_{k+1})$ which realizes the trade-off in (21) satisfies:

$$D_{\bar{m}_\ell}^{(\ell)} \propto n^{1/(2\beta_\ell(1+\sum_{j=1}^{k+1}(2\beta_j)^{-1}))}. \quad (22)$$

Conditions (20) and (22) are satisfied as soon as $\beta_\ell > (k+1)/2$ for every $\ell = 1, \dots, k+1$, and the rate of convergence is $n^{-2\bar{\beta}/(2\bar{\beta}+(k+1))}$ with $\bar{\beta}$ the

harmonic mean. Therefore, the regularity conditions which guarantee the minimax adaptivity get stronger and the rate of convergence decreases as the dimension of the covariate increases.

7.3. Contribution with respect to extant literature

Most estimation procedures developed for current status data are based on the NPMLE originally proposed by [Groeneboom and Wellner \(1992\)](#) and the covariates are usually handled through semi-parametric models (e.g. [Cheng and Wang \(2011\)](#)). Minimax results are assessed under the assumption of a continuous density for the survival and observation times ([van de Geer \(1993\)](#)). Nevertheless, very few adaptive procedures are proposed and they do not include covariates. In alternative to the NPMLE, [Ma and Kosorok \(2006\)](#) and [Brunel and Comte \(2009\)](#) enlighten the parallel between regression and current status data frameworks and propose several adaptive estimation procedures adapted from regression analysis, which exhibit rates of convergence typical of a regression context.

In this paper, we develop the mean-square procedure from [Brunel and Comte \(2009\)](#). Our first contribution is the inclusion of a covariate in a nonparametric adaptive estimation procedure. The second improvement is the computation of the minimax rate over anisotropic Besov balls which assesses the optimality of the regression-type approach in the current status setting; the minimax rate for independent survival and observation times can be derived as a particular case, proving the minimaxicity of the mean-square estimator from [Brunel and Comte \(2009\)](#).

7.4. Perspectives

The parallel between current status and regression frameworks would allow the implementation of other adaptive methods. We briefly present some advantages and drawbacks of alternative approaches that may be considered.

Lepski's method provides locally minimax results up to a logarithm loss unavoidable in pointwise adaptive procedures, and more recent developments display oracle inequalities. Adaptation over classes of regularity is achieved through a bias-variance compromise from kernel estimators ([Lepski \(1991\)](#)). [Goldenshluger and Lepski \(2008\)](#) propose a modification of the original method based on the ordering of the classes of regularity in order to estimate multi-dimensional function in a gaussian regression framework.

Adaptation of this procedure for current status data framework is conceivable, but as for model selection, a specific study of the upper-bound would be necessary. Besides, the implementation of Lepski’s procedure presents a high computational complexity, especially in multi-dimensional context. Since the estimation of the conditional c.d.f. from current status data requires large sample size, more easily computable methods may be preferred. However, Lepski’s method allows local adaptivity which may be favoured when the target function present irregular smoothness.

Wavelet thresholding methods are widely used in practical situations, in particular in signal processing. The procedure consists in a decomposition in a multi-scale wavelet basis followed by a thresholding of the smallest coefficients. The estimators reach the minimax rate of convergence over various classes of regularity, usually up to a logarithm loss (e.g. [Kerkycharian and Picard \(2004\)](#)). [Massart \(2007\)](#) demonstrates the equivalence between wavelet thresholding and the *complete* variable selection procedure which falls into the model selection framework with a collection including a large number of models with the same dimension. Therefore, following the adaptation of tools developed for regression estimation by model selection to the current status framework, wavelet thresholding could be developed in future research.

The mean-square contrast is naturally adapted to the empirical norm and additional assumptions are required for the control of the L^2 -risk. Conversely, quotient regression estimators presented e.g. in [Tsybakov \(2004\)](#) are directly adapted to the L^2 norm. [Brunel and Comte \(2009\)](#) show that the rate of convergence of the model selection quotient estimator from current status data depends on the minimum regularity of the c.d.f. and the observation time density. Therefore, this procedure does not reach the minimax rate if the c.d.f. is smoother than the observation time.

As the conditional c.d.f. lies into $[0, 1]$, procedures for bounded regression may be considered. In particular, we present a comparison of our method with the model selection framework for bounded regression proposed by [Massart \(2007\)](#). The general assumptions are less restrictive than in the classical regression framework since the errors are not supposed to be independent from the design, but the procedure requires models which consist of bounded functions. Two types of bounded models could be directly derived from the linear models $\{S_m, m \in I_n\}$:

- Thresholded linear models: $\tilde{S}_m^{(1)} = \{\tilde{t} = \max(\min(t, 1), 0), t \in S_m\}$
- Restricted linear models: $\tilde{S}_m^{(2)} = \{t \in S_m, t(x) \in [0, 1], \forall x\}$.

We present a brief overview of the advantages and difficulties encountered in the implementation of these models on the interval censoring design. According to Theorem 8.5 in [Massart \(2007\)](#), for each model m , two quantities have to be controlled in order to obtain minimax results: 1/ the stochastic modulus of uniform continuity ϕ_m of the centered empirical process, 2/ the bias term via results of approximation on regularity spaces. On the one hand, the bias term for $\tilde{S}_m^{(1)}$ is directly controlled by results on linear models, but since the unit ball of $\tilde{S}_m^{(1)}$ is larger than the one of S_m , the computation of the function ϕ_m requires a specific study. On the other hand, as $\tilde{S}_m^{(2)}$ is included in S_m , ϕ_m can directly be deduced from results on linear models, but the bias term which involved the L^2 -projection of F on L^∞ -balls does not falls into classical approximation setting.

Besides, an alternative procedure based on finite models, originally developed by [Massart \(2007\)](#) in a Gaussian framework (Chapter 4), could be considered in order to include bounded models. For every $\alpha > 0$, a net of radius α is extracted from a set of function \mathcal{S} including the target function. An appropriate choice of radii $\{\alpha_m, m \in I_n\}$ based on wavelet or polynomial characterization of Besov classes provides minimax estimation.

Thus, our model selection procedure takes advantage of classical tools for the control of the deviation of empirical process on linear models as well as extant results of approximation on Besov spaces, but does not directly fit into the bounded regression framework. Alternative approaches using bounded models could be considered for future research, but give rise to other technical difficulties than the one encountered with linear models.

7.5. Conclusion

The model selection mean-square estimator of the conditional c.d.f. satisfies non-asymptotic adaptivity on a collection of models, as well as minimaxity. The control of the upper-bound takes advantages of the tools used in classical regression. Furthermore, our results underline the optimality of the regression-type approach, which establishes a connection between the interval censoring framework and the widely developed field of regression estimation and makes the way for the adaptation of regression methods to current status design.

8. Proofs

8.1. Talagrand Inequality

The risk of the model selection estimator is controlled with the following concentration inequality based on results from [Talagrand \(1996\)](#).

Theorem 8.1. *Let (V_1, \dots, V_n) be independent random variables, and \mathcal{F} be a countable set of applications from \mathbb{R} to \mathbb{R}^n such that for every $f \in \mathcal{F}$, the function $-f$ defined as $(-f)(x) = -(f(x))$ for every x is in \mathcal{F} as well. Let*

$$Z = \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n (f^{(i)}(V_i) - \mathbb{E}[f^{(i)}(V_i)]) \right|$$

and b , v and \mathbb{H} be such that

$$\sup_{f \in \mathcal{F}} \left(\sup_{i=1, \dots, n} \|f^{(i)}\|_\infty \right) \leq b, \quad \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \text{Var}(f^{(i)}(V_i)) \leq v \quad \text{and} \quad \mathbb{E}Z \leq \mathbb{H}.$$

Then, for every $\theta > 1$, there exists \bar{C} , \bar{C}' , \bar{K} , \bar{K}' such that for every n ,

$$\mathbb{E} [(Z^2 - \theta \mathbb{H}^2)_+] \leq \bar{C} \frac{v}{n} \exp\left(-\bar{K} \frac{n \mathbb{H}^2}{v}\right) + \bar{C}' \frac{b^2}{n^2} \exp\left(-\bar{K}' \frac{n \mathbb{H}}{b}\right)$$

Theorem 8.1 is derived from Theorem 1.1 in [Klein and Rio \(2005\)](#) by setting $s^{(i)}(t) = \frac{1}{b}(f^{(i)}(t) - \mathbb{E}[f^{(i)}(V_i)])$. Similarly to [Birgé and Massart \(1998\)](#), Corollary 2, we get

$$P[Z \geq (1 + \nu)\mathbb{H} + x] \leq \exp\left(-\frac{n}{3} \min\left(\frac{y^2}{2v}, \frac{2 \min(1, \nu)y}{7b}\right)\right).$$

Then we use the formula $\mathbb{E}[X_+] \leq \int_0^{+\infty} P[X \geq s] ds$ to get

$$\mathbb{E} [(Z^2 - \theta \mathbb{H}^2)_+] \leq \int_0^{+\infty} P[Z \geq \sqrt{\theta \mathbb{H}^2 + s}] ds \leq \int_0^{+\infty} P[Z \geq \alpha_1 \mathbb{H} + \sqrt{\alpha_2 \mathbb{H} + \alpha_3 s}] ds$$

for some positive $(\alpha_1, \alpha_2, \alpha_3)$, and a simple integration provides the result of Theorem 8.1.

8.2. Proof of Theorem 4.1.

The proof is similar to Brunel and Comte (2009), Section 6.4, but the conditional empirical norm is considered instead of the L^2 one. This modification entails changes in the proofs that are displayed in this section. The vectors \mathbf{X} and \mathbf{T} are assumed to be fixed along this section. Let $m \in I_n$ and $F_m \in S_m$, by definition of \widehat{F}_m and \widehat{m} ,

$$\begin{aligned} \|\widehat{F}_m - F\|_n^2 &\leq \|F_m - F\|_n^2 + \text{pen}(m) - \text{pen}(\widehat{m}) + 2\nu_n(\widehat{F}_m - F_m) \\ &\leq \|F_m - F\|_n^2 + \text{pen}(m) - \text{pen}(\widehat{m}) + \|\widehat{F}_m - F_m\|_n \sup_{t \in S_m + S_{\widehat{m}}, \|t\|_n \leq 1} |\nu_n(t)| \end{aligned}$$

with ν_n defined in (9). Let $\theta' = \theta^{1/3} > 1$, and $p(m, m') = \theta'(D_m + D_{m'})/(4n)$. We recall that for every $(x, y) \in \mathbb{R}^2$, $2xy \leq x^2/\theta'^2 + \theta'^2 y^2$ and $(x + y)^2 \leq \theta' x^2 + (1 + 1/(\theta' - 1))y^2$. Therefore,

$$\begin{aligned} \|\widehat{F}_m - F\|_n^2 &\leq \|F_m - F\|_n^2 + \text{pen}(m) - \text{pen}(\widehat{m}) + \theta'^2 p(m, \widehat{m}) + \frac{1}{\theta'^2} \|\widehat{F}_m - F_m\|_n^2 \\ &\quad + \theta'^2 \sup_{t \in S_m + S_{\widehat{m}}, \|t\|_n \leq 1} ((\nu_n(t))^2 - p(m, \widehat{m})) \\ &\leq \|F_m - F\|_n^2 + 2\text{pen}(m) \\ &\quad + \frac{1}{\theta'^2} \left(\|\widehat{F}_m - F\|_n^2 + \theta' \left(1 + \frac{1}{\theta' - 1} \right) \|F_m - F\|_n^2 \right) \\ &\quad + \theta'^2 \sum_{m' \in I_n} \sup_{t \in S_m + S_{m'}, \|t\|_n \leq 1} ((\nu_n(t))^2 - p(m, m')) \end{aligned}$$

The end of the proof is provided by Theorem 8.1. According to the comment in Section 3.2, the supremum of ν_n over the $\|\cdot\|_n$ -unit ball of $S_m + S_{m'}$ is equal to the supremum over the unit ball $\mathcal{B}_{m+m'}^*$ of $(S_m + S_{m'})^*$. Let $\{\varphi_\lambda, \lambda \in J_{m+m'}^*\}$ be a $\|\cdot\|_n$ -orthogonal basis of $(S_m + S_{m'})^*$, by Cauchy-Schwartz Inequality and equation (10),

$$\mathbb{E} \left[\sup_{t \in \mathcal{B}_{m+m'}^*} (\nu_n(t))^2 |(\mathbf{X}, \mathbf{T}) \right] \leq \mathbb{E} \left[\sup_{\lambda \in J_{m+m'}^*} (\nu_n(\varphi_\lambda))^2 |(\mathbf{X}, \mathbf{T}) \right] \leq \frac{D_m + D_{m'}}{4n} = \mathbb{H}^2. \quad (23)$$

Moreover, for every $t \in \mathcal{B}_{m+m'}^*$ and $i \in \{1, \dots, n\}$,

$$|\delta_i t(X_i, T_i)| \leq |t(X_i, T_i)| \leq \sqrt{n} \|t\|_n \leq \sqrt{n} = b. \quad (24)$$

Finally, with a similar computation as (10), the supremum of the variance process is upper bounded by $v = 1/4$. Therefore, Theorem 8.1 with the values of \mathbb{H} , b and v computed above concludes the proof of Theorem 4.1. \square

8.3. Proof of Theorem 4.2

The proof is based on results presented by Baraud (2002) in a uni-dimensional context, but the demonstrations are identical for two-dimensional functions. Let $\rho_0 > h_0^{-1}$ and

$$\Omega_n = \left\{ \frac{\|t\|_n^2}{\|t\|^2} \leq \rho_0, \forall t \in S_n \right\}.$$

On the one hand, according to Lemma 5.1 in Baraud (2002),

$$\mathbb{E} \left[\|\tilde{F}_{\hat{m}} - F\|_{\mathbb{I}_{\Omega_n}} \right] \leq C \inf_{t \in S_n} \|F - t\|^2 + \mathbb{E} \left[\|\tilde{F}_{\hat{m}} - F\|_n^2 \right] \quad (25)$$

for some constant C . Moreover, by integrating (12) in Theorem 4.1, we obtain:

$$\mathbb{E} \left[\|\tilde{F}_{\hat{m}} - F\|_n^2 \right] \leq C_1 \inf_{m \in I_n} \left\{ \inf_{t \in S_m} h_1 \|F - t\|^2 + \text{pen}(m) \right\} + \frac{C_2}{n}. \quad (26)$$

Thus, as $\inf_{t \in S_n} \|F - t\|^2 \leq \inf_{t \in S_m} \|F - t\|^2$ for all $m \in I_n$, inequalities (25) and (26) entail:

$$\mathbb{E} \left[\|\tilde{F}_{\hat{m}} - F\|_{\mathbb{I}_{\Omega_n}} \right] \leq C' \inf_{m \in I_n} \left\{ \inf_{t \in S_m} \|F - t\|^2 + \text{pen}(m) \right\} + \frac{C_2}{n}. \quad (27)$$

On the other hand, according to Proposition 4.2 in Baraud (2002),

$$\mathbb{P} [\Omega_n^c] \leq \frac{C''}{n}. \quad (28)$$

Moreover, $\tilde{F}_{\hat{m}}$ and F lie into $[0, 1]$, therefore

$$\|\tilde{F}_{\hat{m}} - F\| \leq h_0^{-1} \|\tilde{F}_{\hat{m}} - F\|_{f_{X,T}} \leq h_0^{-1} \quad a.s. \quad (29)$$

Equations (27), (28) and (29) conclude the proof of Theorem 4.2. \square

8.4. Proof of Theorem 5.1

The proof is based on the following theorem and lemma presented in [Tsybakov \(2004\)](#) (Theorem 2.5 and Lemma 2.7 in Chapter 2). Let $\mathcal{B} = \mathcal{B}_{2,\infty}^\beta(A, L)$. Denote by $K(P, Q)$ the Kullback distance between the distributions P and Q :

$$K(P, Q) = \begin{cases} \int \log(dP/dQ)dP & \text{if } P \ll Q \\ +\infty & \text{otherwise} \end{cases}$$

Theorem 8.2. *Assume that there exist $M \geq 2$ and F_0, \dots, F_M such that*

1. $F_j \in \mathcal{B}$ for every $j \in \{0, \dots, M\}$.
2. $\|F_j - F_l\|^2 \geq 2r$ for every $j \neq l \in \{0, \dots, M\}$.
3. $P_j^{(n)} \ll P_0^{(n)}$ for every $j \in \{0, \dots, M\}$, where $P_j^{(n)}$ denotes the distribution of $(X_i, T_i, \delta_i)_{i=1, \dots, n}$ if $F = F_j$, and for some $0 < \alpha < 1/8$

$$\frac{1}{M} \sum_{j=1}^M K(P_j^{(n)}, P_0^{(n)}) \leq \alpha \log M.$$

Then there exists a constant c such that

$$\inf_{\hat{F}_n} \sup_{F \in \mathcal{B}} \mathbb{E} \left[r \|\hat{F}_n - F\|^2 \right] \geq c.$$

Lemma 8.1. *Let m be a positive integer. There exists a family $(\varepsilon^{(0)}, \dots, \varepsilon^{(M)}) \subset \{0, 1\}^{2^m}$ with $\varepsilon^{(0)} = (0, \dots, 0)$ such that $\log(M) \geq (\log 2/8)2^m$, and*

$$\bar{\rho}(\varepsilon^{(i)}, \varepsilon^{(i')}) \geq \frac{2^m}{8}, \quad \forall i \neq i' \in \{0, \dots, M\}$$

with $\bar{\rho}$ denoting the following distance:

$$\bar{\rho}(\varepsilon, \varepsilon') = \sum_{k=1}^{2^m} \mathbb{1}_{\{\varepsilon_k \neq \varepsilon'_k\}}. \tag{30}$$

Let us present the outline of the proof. Up to rescaling and translation, we assume that $A = [0, 1] \times [0, 1]$. We build a set of c.d.f.:

$$F_\varepsilon = F_0 + \sqrt{\frac{b}{n}} \sum_{S \in R_J} \varepsilon_S \psi_{J,S}$$

where F_0 is a baseline c.d.f., $\varepsilon_S \in \{0, 1\}$ and the $\{\psi_{J,S}, S \in R_J\}$ are wavelets with disconnected supports included in $[0, 1]$, defined from a mother wavelet ψ .

$$\psi_{J,S}(x, u) = 2^{(j_1+j_2)/2} \psi(2^{j_1}x - s_1) \psi(2^{j_2}u - s_2) \quad (31)$$

for some $J = (j_1, j_2) \in \mathbb{N}^2$ and $R_J \subset \mathbb{Z}^2$. We show that

- $\|F_\varepsilon - F_{\varepsilon'}\|^2 = (b/n) \bar{\rho}(\varepsilon, \varepsilon')$.
- $K(P_\varepsilon^{(n)}, P_0^{(n)}) \leq b' 2^{j_1+j_2}$.

Therefore, for every J , Lemma 8.1 ensures the existence of a set $(F_{\varepsilon^{(0)}}, \dots, F_{\varepsilon^{(M)}})$ such that conditions (2) and (3) in Theorem 8.2 are satisfied with an appropriate choice of b . Moreover, we show that $\|F_\varepsilon\|_{\mathcal{B}_{2,\infty}^\beta([0,1]^2)}$ has order

$$\frac{2^{(j_1+j_2)/2} (2^{j_1\beta_1} + 2^{j_2\beta_2} + 1)}{\sqrt{n}} \quad (32)$$

Then J is chosen in order to guarantee that (32) is upper bounded by a constant, which leads to the rate of convergence $r = 2^{j_1+j_2} \propto n^{-\bar{\beta}/(\bar{\beta}+1)}$. We now present the detailed proof.

8.4.1. Construction of the (F_i) 's.

Let

$$F_0(x, u) = \mathbb{I}_{[0,1]}(x) (a \mathbb{I}_{[0,+\infty]}(u) + au \mathbb{I}_{[0,1]}(u) + (1-a) \mathbb{I}_{(1,+\infty]}(u))$$

with $a = \min(1/3, L/2)$. For every $x \in [0, 1]$,

- $F_0(x, u) = 0, \quad \forall u < 0,$
- $F_0(x, u) = 1, \quad \forall u > 1,$
- $F_0(x, \cdot)$ is increasing on $[0, 1]$ and $F_0(x, u) \in [a, 2a] \subset (0, 1)$ for every $u \in [0, 1]$,

thus F_0 is a conditional distribution. Let ψ be a one-dimensional wavelet supported on $[0, 1]$. Let $J = (j_1, j_2)$ be a couple of non-negative integers determined further. For every $S = (s_1, s_2) \in \mathbb{Z}^2$, let $\psi_{J,S}$ be defined in (31). There exists a subset R_J of \mathbb{Z}^2 such that

- $Supp(\psi_{J,S}) = I_{J,S} \subset]0, 1[^2$ for every $S \in R_J$,
- The applications $\{\psi_{J,S}, S \in R_J\}$ have disjoint supports ,
- $|R_J| = 2^{j_1+j_2}$.

Let b be a positive constant which will be determined later. For every $\varepsilon \in \{0, 1\}^{|R_J|}$, let

$$G_\varepsilon = \sqrt{\frac{b}{n}} \sum_{S \in R_J} \varepsilon_S \psi_{J,S}$$

and $F_\varepsilon = F_0 + G_\varepsilon$. For every $x \in [0, 1]$,

- $F_\varepsilon(x, u) = F_0(x, u) = 0, \quad \forall u < 0$
- $F_\varepsilon(x, u) = F_0(x, u) = 1 \quad \forall u > 1$

Moreover let $(x, u) \in [0, 1]^2$,

$$F_\varepsilon(x, u) = a + \int_0^u \left(a + \sqrt{\frac{b}{n}} \sum_{S \in R_J} \varepsilon_S \frac{\partial \psi_{J,S}}{\partial y}(x, y) \right) dy. \quad (33)$$

Assume that

$$\sqrt{\frac{b}{n}} 2^{j_1/2} 2^{3j_2/2} \|\psi\|_\infty \|\psi'\|_\infty \leq \frac{a}{2}. \quad (34)$$

Let $y \in [0, u]$ and S_0 be such that $(x, y) \in I_{J,S_0}$

$$\left| \sqrt{\frac{b}{n}} \sum_{S \in R_J} \varepsilon_S \frac{\partial \psi_{J,S}}{\partial y}(x, y) \right| = \left| \sqrt{\frac{b}{n}} \varepsilon_{S_0} \frac{\partial \psi_{J,S_0}}{\partial y}(x, y) \right| < \frac{a}{2}.$$

Therefore the term in the integral in (33) is positive and the application $F_\varepsilon(x, \cdot)$ is increasing on $[0, 1]$. Moreover, as $\psi_{J,S}(x, 1) = 0$ for every $S \in R_J$, $F_\varepsilon(x, 1) = F_0(x, 1) = 2a < 1$. Thus F_ε is a conditional distribution function on $[0, 1]^2$.

8.4.2. *Condition which guarantees that $F_\varepsilon \in \mathcal{B}$ for every ε .*

On the one hand, assume that ψ is regular enough, then according to Hochmuth (2002) (Theorem 3.5),

$$|G_\varepsilon|_{\mathcal{B}_{2,\infty}^\beta([0,1]^2)} \leq (2^{j_1\beta_1} + 2^{j_2\beta_2}) \|G_\varepsilon\|.$$

Moreover, as the $\{\psi_{J,S}, S \in R_J\}$ have disjoint supports,

$$\|G_\varepsilon\|^2 = \frac{b}{n} \left\| \sum_{S \in R_J} \varepsilon_S \psi_{J,S} \right\|^2 = \frac{b}{n} \sum_{S \in R_J} \varepsilon_S^2 \|\psi_{J,S}\|^2.$$

By definition of the wavelets, $\|\psi_{J,S}\| = \|\psi\| = 1$, hence

$$\|G_\varepsilon\| \leq \sqrt{\frac{b}{n}} |R_J| = \sqrt{\frac{b}{n}} 2^{(j_1+j_2)/2}.$$

Thus

$$\|G_\varepsilon\|_{\mathcal{B}_{2,\infty}^\beta([0,1]^2)} = |G_\varepsilon|_{\mathcal{B}_{2,\infty}^\beta([0,1]^2)} + \|G_\varepsilon\| \leq \sqrt{\frac{b}{n}} 2^{(j_1+j_2)/2} (2^{j_1\beta_1} + 2^{j_2\beta_2} + 1).$$

On the other hand, $|F_0|_{\mathcal{B}_{2,\infty}^\beta([0,1]^2)} = 0$. Indeed, let $r_i = \lfloor \beta_i \rfloor + 1$ for $i = 1$ and 2 . Then $r_1 \geq 1$, $r_2 \geq 2$ and

$$|F_0|_{\mathcal{B}_{2,\infty}^\beta([0,1]^2)} = \sup_{t>0} [t^{-\beta_1} \omega_{r_1,1}(F_0, t, [0, 1]^2)_2 + t^{-\beta_2} \omega_{r_2,2}(F_0, t, [0, 1]^2)_2].$$

Besides let $h > 0$ and

$$\Omega_{h,1}^{r_1} = \{(x, u) \in [0, 1]^2, (x + r_1 h, u) \in [0, 1]^2\}.$$

For every $(x, u) \in \Omega_{h,1}^{r_1}$ $F_0(x + h, u) = F_0(x, u)$. So, as $r_1 \geq 1$, $\Delta_{h,1}^{r_1} F_0(x, u) = 0$. Hence

$$\omega_{r_1,1}(F_0, t, [0, 1]^2)_2 = \sup_{|h| \leq t} \|\Delta_{h,1}^{r_1} F_0\|_{L^2(\Omega_{h,1}^{r_1})} = 0.$$

Moreover, for every $(x, u) \in [0, 1]^2$

$$\Delta_{h,2}^1 F_0(x, u) = ah \quad \Rightarrow \quad \Delta_{h,2}^{r_2} F_0(x, u) = \Delta_{h,2}^{r_2-1} \Delta_{h,2}^1 F_0(x, u) = 0$$

as $r_2 - 1 \geq 1$. Then $\omega_{r_2,2}(F_0, t, [0, 1]^2)_2 = 0$ and consequently $|F_0|_{\mathcal{B}_{2,\infty}^\beta([0,1]^2)} = 0$. Moreover

$$\|F_0\|_{\mathcal{B}_{2,\infty}^\beta([0,1]^2)} = \sqrt{\int_0^1 \int_0^1 a^2(1+u)^2 dudx} = \sqrt{\frac{7}{3}}a$$

and

$$\|F_\varepsilon\|_{\mathcal{B}_{2,\infty}^\beta([0,1]^2)} \leq \|F_0\|_{\mathcal{B}_{2,\infty}^\beta([0,1]^2)} + \|G_\varepsilon\|_{\mathcal{B}_{2,\infty}^\beta([0,1]^2)} \leq \sqrt{\frac{7}{3}}a + \sqrt{\frac{b}{n}}2^{(j_1+j_2)/2}(2^{j_1\beta_1} + 2^{j_2\beta_2} + 1).$$

By definition $a \leq L/2$ so $\|F_\varepsilon\|_{\mathcal{B}_{2,\infty}^\beta([0,1]^2)} \leq L$ as soon as

$$\sqrt{\frac{b}{n}}2^{(j_1+j_2)/2}(2^{j_1\beta_1} + 2^{j_2\beta_2} + 1) \leq L \left(1 - \sqrt{\frac{7}{12}}\right). \quad (35)$$

8.4.3. *Expression of $\|F_\varepsilon - F_{\varepsilon'}\|^2$.*

$$\|F_\varepsilon - F_{\varepsilon'}\|^2 = \frac{b}{n} \sum_{S \in R_J} \int_{I_{J,S}} (\varepsilon_S - \varepsilon'_S)^2 \psi_{J,S}^2(x, u) dx du = \frac{b}{n} \sum_{S \in R_J} \mathbb{1}_{\{\varepsilon_S \neq \varepsilon'_S\}} = \frac{b}{n} \bar{\rho}(\varepsilon, \varepsilon'). \quad (36)$$

8.4.4. *Upper bound of $K(P_\varepsilon^{(n)}, P_0^{(n)})$*

For every $i \in \{1, \dots, n\}$, under F_ε , (X_i, T_i, δ_i) has density

$$p_\varepsilon(x, u, d) = [(F_\varepsilon(x, u))^d (1 - F_\varepsilon(x, u))^{1-d}] f_{(X,T)}(x, u)$$

with respect to $\mathcal{L} \otimes \mathcal{L} \otimes \mu$ where \mathcal{L} is the Lebesgue measure and μ is the counting measure on \mathbb{N} . Similarly, under F_0 , (X_i, T_i, δ_i) has density

$$p_0(x, u, d) = [(F_0(x, u))^d (1 - F_0(x, u))^{1-d}] f_{(X,T)}(x, u)$$

with respect to $\mathcal{L} \otimes \mathcal{L} \otimes \mu$. For every $\varepsilon \in \{0, 1\}^{|R_J|}$, P_ε is absolutely continuous with respect to P_0 . Indeed,

$$\begin{aligned} F_0(x, u) = 0 &\Rightarrow (x, u) \notin [0, 1] \times [0, +\infty[\Rightarrow F_\varepsilon(x, u) = 0, \\ F_0(x, u) = 1 &\Rightarrow (x, u) \in [0, 1] \times [1, +\infty[\Rightarrow F_\varepsilon(x, u) = 1. \end{aligned}$$

Then

$$K(P_\varepsilon, P_0) = \int_{\mathbb{R}^2} \left[\log \left(\frac{F_\varepsilon(x, u)}{F_0(x, u)} \right) F_\varepsilon(x, u) + \log \left(\frac{1 - F_\varepsilon(x, u)}{1 - F_0(x, u)} \right) (1 - F_\varepsilon(x, u)) \right] f_{(X, T)}(x, u) dx du$$

Out of the intervals $\{I_{J, S}, S \in R_J\}$, F_ε and F_0 are equal. Hence

$$\begin{aligned} K(P_\varepsilon, P_0) &= \sum_{S \in R_J} \int_{I_{J, S}} \left[\log \left(1 + \frac{\theta_S}{a(1+u)} \right) (a(1+u) + \theta_S) \right. \\ &\quad \left. + \log \left(1 - \frac{\theta_S}{1 - a(1+u)} \right) (1 - a(1+u) - \theta_S) \right] f_{(X, T)}(x, u) dx du \end{aligned}$$

where $\theta_S = \varepsilon_S \sqrt{b/n} \psi_{J, S}(x, u)$. For every $S \in R_J$ and $(x, u) \in I_{J, S}$

$$\frac{\theta_S}{a(1+u)} = \frac{F_\varepsilon(x, u)}{F_0(x, u)} - 1 > -1 \quad \text{and} \quad -\frac{\theta_S}{1 - a(1+u)} = \frac{1 - F_\varepsilon(x, u)}{1 - F_0(x, u)} - 1 > -1$$

Noting that $\log(1+v) \leq v$ for every $v > -1$ we obtain

$$K(P_\varepsilon, P_0) \leq \sum_{S \in R_J} \int_{I_{J, S}} \left[\theta_S + \frac{\theta_S^2}{a(1+u)} - \theta_S + \frac{\theta_S^2}{1 - a(1+u)} \right] f_{(X, T)}(x, u) dx du.$$

For every $u \in [0, 1]$,

$$0 < \frac{1}{a(1+u)} \leq \frac{1}{a} \quad \text{and} \quad 0 < \frac{1}{1 - a(1+u)} \leq \frac{1}{1 - 2a}.$$

Thus

$$\begin{aligned} K(P_\varepsilon, P_0) &\leq \left(\frac{1}{a} + \frac{1}{1 - 2a} \right) \sum_{S \in R_J} \int_{I_{J, S}} \theta_S^2 f_{(X, T)}(x, u) dx du \\ &\leq \left(\frac{1}{a} + \frac{1}{1 - 2a} \right) \frac{b}{n} \|f_{(X, T)}\|_\infty |R_J| = a'b \|f_{(X, T)}\|_\infty \frac{2^{j_1 + j_2}}{n}. \end{aligned}$$

where $a' = 1/a + 1/(1 - 2a)$. Finally,

$$K(P_\varepsilon^{(n)}, P_0^{(n)}) \leq a'b \|f_{(X, T)}\|_\infty 2^{j_1 + j_2}.$$

8.4.5. Conclusion

According to Lemma 8.1, there exists a family $(\varepsilon^{(0)}, \dots, \varepsilon^{(M)}) \subset \{0, 1\}^{|R_J|}$ with $\varepsilon^{(0)} = (0, \dots, 0)$ such that $\log(M) \geq (\log 2/8)2^{j_1+j_2}$ and

$$\bar{\rho}(\varepsilon^{(i)}, \varepsilon^{(i')}) \geq \frac{|R_J|}{8} = \frac{2^{j_1+j_2}}{8}, \quad \forall i \neq i' \in \{0, \dots, M\}.$$

Now the parameters B_0 , b , j_1 and j_2 are chosen so that the family $(F_{\varepsilon^{(0)}}, \dots, F_{\varepsilon^{(M)}})$ satisfies the assumptions of Theorem 8.2 with

$$r = B_0 n^{\bar{\beta}/(\bar{\beta}+1)}.$$

Let

$$b = \frac{\log 2}{72 \|f_{(X,T)}\|_{\infty} a'}, \quad c_0 = \left[\frac{L}{4\sqrt{b}} \left(1 - \sqrt{\frac{7}{12}} \right) \right]^{1/(1+\beta_1+\beta_2)} \quad \text{and} \quad B_0 = bc_0^2/32.$$

Let j_1 and j_2 be in \mathbb{N}^* such that

$$\begin{aligned} (c_0/2)n^{\beta_2/(\beta_1+\beta_2+2\beta_1\beta_2)} &\leq 2^{j_1} \leq c_0 n^{\beta_2/(\beta_1+\beta_2+2\beta_1\beta_2)} \\ (c_0/2)n^{\beta_1/(\beta_1+\beta_2+2\beta_1\beta_2)} &\leq 2^{j_2} \leq c_0 n^{\beta_1/(\beta_1+\beta_2+2\beta_1\beta_2)}. \end{aligned} \quad (37)$$

The existence of j_1 and j_2 is guaranteed for n larger than an integer n_0 depending on (c_0, β) . Then for every $i, i' \in \{0, \dots, M\}$

$$\begin{aligned} \|F_{\varepsilon^{(i)}} - F_{\varepsilon^{(i')}}\|^2 &\geq \frac{b}{n} \frac{2^{j_1+j_2}}{8} \geq \frac{bc_0^2}{32n} n^{(\beta_1+\beta_2)/(\beta_1+\beta_2+2\beta_1\beta_2)} \\ &= B_0 n^{-2\beta_1\beta_2/(\beta_1+\beta_2+2\beta_1\beta_2)} = B_0 n^{-\bar{\beta}/(\bar{\beta}+1)} \end{aligned} \quad (38)$$

hence condition (2) in Theorem 8.2 is satisfied.

Moreover

$$\frac{1}{M} \sum_{l=0}^M K(P_{\varepsilon^{(l)}}^{(n)}, P_0^{(n)}) \leq a' \|f_{(X,T)}\|_{\infty} b 2^{j_1+j_2} = \frac{\log 2}{72} 2^{j_1+j_2} \leq \frac{\log M}{9} \quad (39)$$

hence condition (3) in Theorem 8.2 is satisfied with $\alpha = 1/9$.

Finally condition (1) in Theorem 8.2 is satisfied as soon as (34) and (35) hold. Besides, $\beta_1 > 0$ and $\beta_2 > 1$ and by (37) j_1 and j_2 are increasing with

n . Therefore $2^{(j_1+j_2)/2}(2^{j_1\beta_1} + 2^{j_2\beta_2} + 1)$ increases faster than $2^{j_1/2}2^{3j_2/2}$ and for n larger than an integer n_1 depending on ψ and L , (34) holds as soon as (35) holds. Moreover (35) holds as soon as

$$\sqrt{bc_0}n^{-\beta_1\beta_2/(\beta_1+\beta_2+2\beta_1\beta_2)} \left((c_0^{\beta_1} + c_0^{\beta_2})n^{\beta_1\beta_2/(\beta_1+\beta_2+2\beta_1\beta_2)} + 1 \right) \leq L \left(1 - \sqrt{\frac{7}{12}} \right)$$

which is ensured if

$$\sqrt{bc_0}(c_0^{\beta_1} + c_0^{\beta_2}) \leq \frac{L}{2} \left(1 - \sqrt{\frac{7}{12}} \right) \quad \text{and} \quad (40)$$

$$\sqrt{bc_0}n^{-\beta_1\beta_2/(\beta_1+\beta_2+2\beta_1\beta_2)} \leq \frac{L}{2} \left(1 - \sqrt{\frac{7}{12}} \right). \quad (41)$$

On the one hand (40) holds as soon as

$$2c_0^{\beta_1+\beta_2+1} \leq \frac{L}{2\sqrt{b}} \left(1 - \sqrt{\frac{7}{12}} \right)$$

which is guaranteed by the definition of c_0 . On the other hand there exists an integer n_2 depending on (β, c_0) such that (41) is satisfied for every $n \geq n_2$.

Thus for every $n \geq \max(n_0, n_1, n_2)$, conditions (1), (2) and (3) in Theorem 8.2 hold with $r = B_0n^{-\bar{\beta}/(\bar{\beta}+1)}$, which concludes the proof of Theorem 5.1. \square

8.4.6. Minimax rate of convergence in the absence of covariate

For sake of simplicity, we keep the same notations. Let $(Y_i, T_i)_{i=1, \dots, n}$ be an i.i.d. sample with T_i and Y_i independent positive random variable. Assume that a sample $(T_i, \delta_i)_{i=1, \dots, n}$ is observed with $\delta_i = \mathbb{1}_{\{Y_i \leq T_i\}}$. Let $F(y) = P[Y_i \leq y]$ the c.d.f. of Y_i . Let $\beta > 1$, $L > 0$ and $\mathcal{B}_{2, \infty}^\beta([0, 1], L)$ the one-dimensional Besov ball of regularity β and radius L . Then there exists a constant c such that

$$\inf_{\widehat{F}_n} \sup_{F \in \mathcal{B}_{2, \infty}^\beta([0, 1], L)} \mathbb{E} \left[r \|\widehat{F}_n - F\|^2 \right] \geq c.$$

The proof is identical to the proof of Theorem 5.1 except that the dependence with respect to X is omitted. More precisely, let $a = \min(1/3, L/2)$ and

$$F_0(u) = a\mathbb{1}_{[0, +\infty[}(u) + au\mathbb{1}_{[0, 1]}(u) + (1 - a)\mathbb{1}_{(1, +\infty[}(u)$$

for every $u \in \mathbb{R}$. Let

$$b = \frac{\log 2}{72 \|f_T\|_\infty a'}, \quad c_0 = \left[\frac{L}{2\sqrt{b}} \left(1 - \sqrt{\frac{7}{12}} \right) \right]^{1/(1+\beta)} \quad \text{and} \quad B_0 = bc_0/16$$

with $a' = (1/a) + (1/(1-2a))$ and f_T the density of T_i . Let $j \in \mathbb{N}^*$ be such that $(c_0/2)n^{1/(2\beta+1)} \leq 2^j \leq c_0 n^{1/(2\beta+1)}$. For $s \in \mathbb{N}^*$, let $\psi_{j,s}(u) = 2^{j/2} \psi(2^j u - s)$. Let R_j be a subset of \mathbb{Z} such that:

- $\text{Supp}(\psi_{j,s}) \subset [0, 1], \forall s \in R_j$.
- The functions $\{\psi_{j,s}, s \in R_j\}$ have disjoint supports.
- $|R_j| = 2^j$.

Let $M \in \mathbb{N}^*$ and $(\varepsilon^{(0)}, \dots, \varepsilon^{(M)}) \in \{0, 1\}^{|R_j|}$ be such that $\log(M) \geq (\log 2/8)2^j$ and

$$\bar{\rho}(\varepsilon^{(i)}, \varepsilon^{(i')}) \geq \frac{|R_j|}{8}, \quad \forall i \neq i' \in \{0, \dots, M\}.$$

For every $i \in \{0, \dots, M\}$, let

$$F_{\varepsilon^{(i)}} = F_0 + \sqrt{\frac{b}{n}} \sum_{s \in R_j} \varepsilon_s^{(i)} \psi_{j,s}.$$

Then:

- Similarly to (39),

$$\frac{1}{M} \sum_{l=0}^M K(P_{\varepsilon^{(l)}}^{(n)}, P_0^{(n)}) \leq \frac{\log M}{9}$$

- Similarly to equation (38), for every $i, i' \in \{0, \dots, M\}$

$$\|F_{\varepsilon^{(i)}} - F_{\varepsilon^{(i')}}\|^2 \geq B_0 n^{-2\beta/(2\beta+1)}$$

- Similarly to (34) and (35), $F_{\varepsilon^{(i)}}$ is a c.d.f. and lies in $\mathcal{B}_{2,\infty}^\beta([0, 1], L)$ as soon as

$$\begin{aligned}
\text{(a)} \quad & \sqrt{\frac{b}{n}} 2^{3j/2} \|\psi'\|_\infty \leq \frac{a}{2} \\
\text{(b)} \quad & \sqrt{\frac{b}{n}} 2^{j/2} (2^{j\beta} + 1) \leq L \left(1 - \sqrt{\frac{7}{12}} \right)
\end{aligned}$$

Since $\beta > 1$, (a) holds as soon as (b) is satisfied and (b) is guaranteed by the definition of c_0 . Thus, the conditions of Theorem 8.2 are fulfilled for r proportional to $n^{-2\beta/(2\beta+1)}$. \square

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