Connected components for planar graphs

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Plan

1. Definition of the Tutte polynomial
   - Connexion with random cluster partition function
   - On derivatives of the Tutte polynomial
   - Particular cases of the Tutte polynomial
   - For planar graphs
2. Some results on connected components
   - Applications : finite planar graphs
   - Applications : infinite planar graphs
   - Perspectives

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Definition of the Tutte polynomial

$G = (V, E)$ a connected planar graph, V ($|v| = n$) the set of vertices, E ($|E| = m$) the set of edges.

Definition

The Tutte polynomial of G denoted by $T(G, x, y)$ can be obtained by computing the four following rules :
(1) If G has no edges, then $T(G, x, y) = 1$.
(2) If e is an edge of G that is neither a loop nor an isthmus, then

$$T(G, x, y) = T(G'_e, x, y) + T(G''_e, x, y)$$

where G'_e is the graph G with the edge e deleted and G''_e is the graph G with the edge e contracted.
(3) If e is an isthmus, then $T(G, x, y) = xT(G'_e, x, y)$
(4) If e is a loop, then $T(G, x, y) = yT(G'_e, x, y)$.

Example

Construction of the tutte polynomial for a cycle C_n of length n;

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Connexion with random cluster partition function

$$T(G,x,y) = \frac{1}{(x-1)(y-1)^n} \sum_{A \subset E} (y-1)^{|A|}[(x-1)(y-1)]^{k(G,A)}$$

Two important values of the Tutte polynomial

(1) On the curve $(x-1)(y-1) = 1$, ($x = 1/p$ and $y = 1/(1-p)$),

$$T(G, 1/p, 1/(1-p) = (1-p)^{n-m-1} p^{1-n}.$$

(2) On the curve $(x-1)(y-1) = q$, ($x = 1 + q(1-p)/p$ and $y = 1/(1-p)$),

$$\begin{aligned} T(G,x,y) &= \frac{(1-p)^{n-m-1} p^{1-n}}{q} \sum_{A \subset E} p^{|A|} (1-p)^{m-|A|} q^{k(G,A)} \\ &= \frac{T(G, 1/p, 1/(1-p))}{q} E(q^{k(G,p)}) \end{aligned}$$

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On derivatives of the Tutte polynomial

Taking $q = 1$, the factorial moments are

$$E[(k-1)(k-2)\dots(k-i)] = \left(\frac{1-p}{p}\right)^i \frac{\frac{\partial^i}{\partial x^i} T(G, \frac{1}{p}, \frac{1}{1-p})}{T(G, \frac{1}{p}, \frac{1}{1-p})}$$

$$E(k(G,p)) = 1 + \left(\frac{1-p}{p}\right) \frac{\frac{\partial}{\partial x} T(G, \frac{1}{p}, \frac{1}{1-p})}{T(G, \frac{1}{p}, \frac{1}{1-p})}$$

$$\sum_{i=1}^{n} E(|C_i|^{-1}) = E(k(G,p))$$

The expected value of length of the MST of G

$$E(L_{MST}(G)) = \int_0^1 E(k(G,p))\, dp - 1$$

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Particular cases of the Tutte polynomial

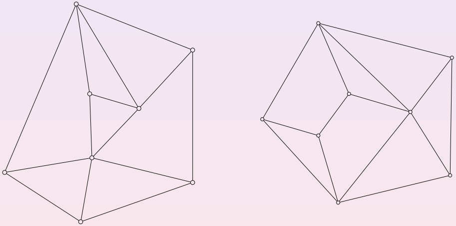
The chromatic polynomial

$$P(G,q) = q^{k(G)}(-1)^{k(G)+n(G)}T(G, x = 1 - q, y = 0)$$

$T(G,1,1)$ number of spanning trees
$T(G,2,1)$ number of spanning forests, (or independent sets)
$T(G,2,2)$ number of spanning subgraphs
$T(G,1,2)$ number of spanning connected subgraphs
$T(G,2,0)$ number of acyclic orientations
$T(G,0,2)$ number of totally cyclic orientations

Be careful

(1) $G^\star$ dual of $G \Rightarrow T(G^\star, x, y) = T(G, y, x)$.
$G = G^\star \Rightarrow T(G, x, y) = T(G, y, x)$ Wrong converse.
(2) $T(G, x, y) = T(H, x, y) \not\Rightarrow G = H$.



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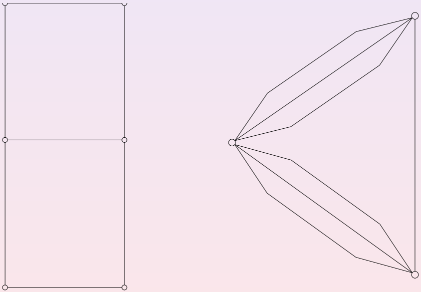
Definition of the dual graph $G^\star$ of a planar graph G

Euler relation : $f = m - n + \alpha$.
$n^\star = f + 1$, $m^\star = m$ and $f^\star = n - 1$.



Some results on connected components

Distribution on edges of G : edge white with probability p.
edge black with probability $1-p$

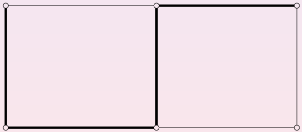
A : set of white edges in G.

$k(G, A, p)$: number of white connected components in (V, A).

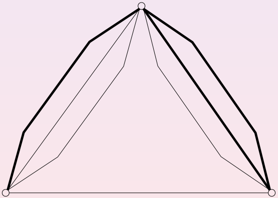
Distribution on edges of $G^\star$: edge white with probability $1-p$.
edge black with probability p

$A^\star$: set of white edges in $G^\star$.

$k(G^\star, A^\star, 1-p)$: number of white connected components in $(V^\star, A^\star)$.



$A \subset E$



$A^\star \subset E^\star$

Some results on connected components

Proposition

Let be G a connected planar graph and denote by $G^\star$ the dual graph of G. For all subset A of E and corresponding subset $A^\star$ of $E^\star$,

$$k(G^\star, A^\star, 1-p) - k(G, A, p) = |A| - n + 1$$

where $|A|$ represents number of elements of A.

Proposition

$$\mathrm{E}(\mathrm{k}(\mathrm{G}^\star, 1-\mathrm{p})) - \mathrm{E}(\mathrm{k}(\mathrm{G}, \mathrm{p})) = \mathrm{mp} - \mathrm{n} + 1 \rightarrow p = \frac{n-1}{m} \ ?$$

Some results on connected components

Proposition

$$\mathrm{E}(\mathrm{k}(\mathrm{G}^\star, 1-\mathrm{p})) - \mathrm{E}(\mathrm{k}(\mathrm{G}, \mathrm{p})) = \mathrm{p}(1-\mathrm{p})\frac{\mathrm{d}}{\mathrm{dp}} \log \mathrm{T}(\mathrm{G}, \frac{1}{\mathrm{p}}, \frac{1}{1-\mathrm{p}}).$$

For self dual graphs ($G = G^\star$) : choose $p = 1/2$.
$\Rightarrow$ On the curve $(x-1)(y-1) = q = 1$, $(x = 1/p, y = 1/(1-p))$,
$T(G, 2, 2) = \min[T(G, x, y)]$.
On the curve $(x-1)(y-1) = q$, $(x = 1 + q(1-p)/p, y = 1/(1-p))$,
$T(G, 1+\sqrt{q}, 1+\sqrt{q}) = \min[T(G, x, y)]$ obtained with $p = \dfrac{\sqrt{q}}{1+\sqrt{q}}$ the critical probability for q-Potts model ?

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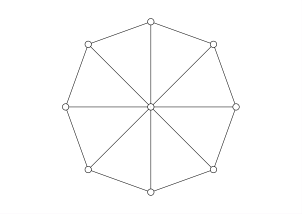
Applications : finite planar graphs

Example : the wheel, a self dual graph.



n vertices, $m = 2(n-1)$ edges.

$$E\left[k(G^{\star}, 1-p)\right] - E\left[k(G,p)\right] = 2(n-1)(p - \frac{1}{2})$$

$\Rightarrow p = 1/2$. The natural choice of p is $1/2$ and so that for all graphs with $m = 2(n-1)$.

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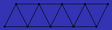
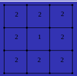
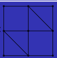
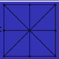
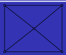
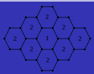
Applications : infinite planar graphs

$\mathrm{E}(\mathrm{k}(\mathrm{G}^*, 1 - \mathrm{p_e})) - \mathrm{E}(\mathrm{k}(\mathrm{G}, \mathrm{p_e})) = \mathrm{mp_e} - \mathrm{n} + 1.$

$p_e = \dfrac{n-1}{m} \Rightarrow \mathrm{E}(\mathrm{k}(\mathrm{G}^*, 1 - \frac{\mathrm{n}-1}{\mathrm{m}})) = \mathrm{E}(\mathrm{k}(\mathrm{G}, \frac{\mathrm{n}-1}{\mathrm{m}})).$

When n tends to infinity with m, m function of n,

$$\Rightarrow \mathrm{E}(\mathrm{k}(\mathrm{G}^\star, 1 - \lim_{\mathrm{n}\to+\infty} \mathrm{p_e})) = \mathrm{E}(\mathrm{k}(\mathrm{G}, \lim_{\mathrm{n}\to+\infty} \mathrm{p_e}).$$

Name	Graph	$p_e = (n-1)/m$	$\lim_{s \to +\infty} p_e$
s squares		$(2s+1)/(3s+1)$	$2/3$
s triangles		$(s+1)/(2s+1)$	$1/2$
Square Lattice		$(2s+1)/(4s)$	$1/2$
Bow-tie Lattice	C2 = 	$(4s)/(10s-3)$	$2/5$
Crossed Matching	C3 = 	$s/(3s-1)$	$1/3$
Octagonal		$s/(3s-1)$	$1/3$
Honeycomb Lattice		$\dfrac{8s^2-3}{12s^2-4s-2}$	$2/3$
kagome lattice		$\dfrac{(6s+1)}{2(6s-1)}$	$1/2$

For honeycomb lattice, $E(k(G,2/3)) = E(k(G^\star,1/3))$.
How to understand for example this $2/3$ associated with honeycomb lattice ?
We have

$$p_e = \frac{n-1}{m} = \frac{2(n-1)}{\sum_{i=1}^{n_{int}} q_i + \sum_{i=1}^{n_{ext}} q_i} = \frac{2}{\frac{1}{n-1}\sum_{i=1}^{n_{int}} q_i + \frac{1}{n-1}\sum_{i=1}^{n_{ext}} q_i}.$$

For all regular lattice with bounded degree,

$$\lim_{n\to+\infty} \frac{n_{ext}}{n} = 0 \Rightarrow \lim_{n\to+\infty} p_e = \lim_{n\to+\infty} 2(n-1)/\sum_{i=1}^{n_{int}} q_i = 2/\bar{q}.$$

Plan

Perspectives

- The special case of self-dual graphs : the location of complex zeros of their Tutte polynomial.
- Trace and determinant for symetric Tutte polynomial.
- Relation between coefficients of the Tutte polynomial.
- The q complex case.
- Symmetry and conformal transformations.