Décomposition par paire pour l'optimisation combinatoire dans les modèles graphiques

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Overview

1. Frame of study
2. Pairwise decomposition : Test and building
3. Experimental results

Formalism

Graphical Model (GM)

A GM is a triplet $(\mathcal{X}, \mathcal{D}, \mathcal{P})$.

- $\mathcal{X} = \{X_1, \ldots, X_n\}$ a set of variables,
- $\mathcal{D} = \{D_{X_1}, \ldots, D_{X_n}\}$ a set of finite domains
- $\mathcal{F} = \{f_1, \ldots, f_e\}$, a set of nonnegative functions, each defined over a subset of variables $\mathbf{S}_i \subseteq \mathcal{X}$ (i.e. the scope) and $\top$ forbidden value

Probabilistic GM

$$\mathbb{P}(\mathcal{X}) \propto \prod_{i=1}^{e} f_i(\mathbf{S}_i)$$

$$(\top = 0)$$

Non probabilistic GM : WCSP
[Schiex, Fargier,Verfaillie, IJCAI95]

$$\texttt{score}(\mathcal{X}) = \sum_{i=1}^{e} f_i(\mathbf{S}_i)$$

$$(\top = +\infty)$$

Optimization task

Probabilistic

The *Most Probable Explanation* (MPE) problem :

$$\underset{\mathcal{X}}{\operatorname{argmax}}\, \mathbb{P}(\mathcal{X})$$

Non probabilistic : WCSP

$$\underset{\mathcal{X}}{\operatorname{argmin}}\, \texttt{score}(\mathcal{X})$$

Equivalence between models

$$\underset{\mathcal{X}}{\operatorname{argmax}}\, \mathbb{P}(\mathcal{X}) = \underset{\mathcal{X}}{\operatorname{argmin}} \sum_{i=1}^{e} -log(f_i(\mathbf{S}_i))$$

Exact methods

- Inference (Variable Elimination : VE, Cluster Tree Elimination)
- Depth First Branch and Bound (DFBB)
- Hybrids
  - DFBB & local reasoning (bounded VE + soft local consistency)

Essential Operators for WCSP

Projection : $f[\mathbf{S}']$

$\mathbf{S}' \subseteq \mathbf{S}$ and $\forall t' \in D_{\mathbf{S}'} f[\mathbf{S}'](t') = \min_{t \in D_{\mathbf{S}} \text{ s.t. } t[\mathbf{S}']=t'} f(t)$

X	Y	f
0	0	5
0	1	7
1	0	6
1	1	4

X	$f[X]$
0	5
1	4

Join : $f = f_1 + f_2$

$f(t) = f_1(t[\mathbf{S}_1]) + f_2(t[\mathbf{S}_2]), \forall t \in D_{\mathbf{S}_1 \cup \mathbf{S}_2}$

Subtraction : $f = f_1 - f_2$

$\mathbf{S}_2 \subseteq \mathbf{S}_1$ and $f(t) = f_1(t) - f_2(t[\mathbf{S}_2]), \forall t \in D_{\mathbf{S}_1}$ with $\top - \top = \top$

i-bounded Variable Elimination (VE(i)) :

elimination of variable X if $\texttt{degree}(X) \leqslant i$

creation of new function : $(\sum_{f_j : X \in \mathbf{S}_j} f_j)\ [\cup_{f_j : X \in \mathbf{S}_j} \mathbf{S}_j\ X]$

soft local consistency :

- Soft Directed Arc Consistency

$$\begin{aligned} f_1(X) &\leftarrow f_1(X) + (f(X,Y) + f_2(Y))[X] \\ f(X,Y) &\leftarrow (f(X,Y) + f_2(Y)) - (f(X,Y) + f_2(Y))[X] \\ f_2(Y) &\leftarrow 0 \end{aligned}$$

Pairwise decomposition

Pairwise decomposition

A *pairwise decomposition* of a cost function $f(\mathbf{S})$ with respect to two variables $X, Y \in \mathbf{S}$ $(X \neq Y)$ is defined by two cost functions $f_1(\mathbf{S} \setminus \{Y\})$ and $f_2(\mathbf{S} \setminus \{X\})$ such that :

$$f(\mathbf{S}) = f_1(\mathbf{S} \setminus \{Y\}) + f_2(\mathbf{S} \setminus \{X\})$$

X	Y	f
0	0	5
0	1	7
1	0	4
1	1	6

=

X	f_1
0	2
1	1

+

Y	f_2
0	3
1	5

Remark Equivalence between pairwise decomposition and pairwise independence in a Gibbs distribution parameterized by f

$$(X \perp\!\!\!\perp Y \mid \mathbf{Z})_f \iff f(X, \mathbf{Z}, Y) = f_1(X, \mathbf{Z}) + f_2(\mathbf{Z}, Y)$$

Pairwise decomposition of $f(\mathbf{S})$

Choose $X, Y \in \mathbf{S}$, $\mathbf{Z} = \mathbf{S} \setminus \{X, Y\}$

1. Is $f(X, \mathbf{Z}, Y)$ pairwise decomposable w.r.t X, Y?

2. How can we build $f_1(X, \mathbf{Z})$ and $f_2(\mathbf{Z}, Y)$?

Testing Pairwise Decomposability (1/3)

$$f(X, \mathbf{Z}, Y) = f_1(X, \mathbf{Z}) + f_2(\mathbf{Z}, Y)$$

$u, v \in D_X$, $\mathbf{z} \in D_{\mathbf{Z}}$, $k, l \in D_Y$

$$\begin{cases} f_1(u, \mathbf{z}) + f_2(\mathbf{z}, l) & = & f(u, \mathbf{z}, l) \\ f_1(v, \mathbf{z}) + f_2(\mathbf{z}, l) & = & f(v, \mathbf{z}, l) \\ f_1(u, \mathbf{z}) + f_2(\mathbf{z}, k) & = & f(u, \mathbf{z}, k) \\ f_1(v, \mathbf{z}) + f_2(\mathbf{z}, k) & = & f(v, \mathbf{z}, k) \end{cases}$$

Testing Pairwise Decomposability (1/3)

$$f(X, \mathbf{Z}, Y) = f_1(X, \mathbf{Z}) + f_2(\mathbf{Z}, Y)$$

$u, v \in D_X$, $\mathbf{z} \in D_{\mathbf{Z}}$, $k, l \in D_Y$

$$\begin{cases} f_1(u, \mathbf{z}) + f_2(\mathbf{z}, l) & = & f(u, \mathbf{z}, l) \\ f_1(v, \mathbf{z}) + f_2(\mathbf{z}, l) & = & f(v, \mathbf{z}, l) \\ f_1(u, \mathbf{z}) + f_2(\mathbf{z}, k) & = & f(u, \mathbf{z}, k) \\ f_1(v, \mathbf{z}) + f_2(\mathbf{z}, k) & = & f(v, \mathbf{z}, k) \end{cases}$$

$$\begin{cases} f_1(u, \mathbf{z}) = f(u, \mathbf{z}, l) & - & f_2(\mathbf{z}, l) & & \\ f_1(v, \mathbf{z}) = f(v, \mathbf{z}, l) & - & f_2(\mathbf{z}, l) & & \\ f_2(\mathbf{z}, k) = f(u, \mathbf{z}, k) & - & f(u, \mathbf{z}, l) & + & f_2(\mathbf{z}, l) \\ f_2(\mathbf{z}, k) = f(v, \mathbf{z}, k) & - & f(v, \mathbf{z}, l) & + & f_2(\mathbf{z}, l) \end{cases}$$

Testing Pairwise Decomposability (1/3)

$$f(X,\mathbf{Z},Y) = f_1(X,\mathbf{Z}) + f_2(\mathbf{Z},Y)$$

$u, v \in D_X,\ \mathbf{z} \in D_{\mathbf{Z}},\ k, l \in D_Y$

$$\begin{cases} f_1(u,\mathbf{z}) + f_2(\mathbf{z},l) & = & f(u,\mathbf{z},l) \\ f_1(v,\mathbf{z}) + f_2(\mathbf{z},l) & = & f(v,\mathbf{z},l) \\ f_1(u,\mathbf{z}) + f_2(\mathbf{z},k) & = & f(u,\mathbf{z},k) \\ f_1(v,\mathbf{z}) + f_2(\mathbf{z},k) & = & f(v,\mathbf{z},k) \end{cases}$$

$$\begin{cases} f_1(u,\mathbf{z}) = f(u,\mathbf{z},l) & - & f_2(\mathbf{z},l) & & \\ f_1(v,\mathbf{z}) = f(v,\mathbf{z},l) & - & f_2(\mathbf{z},l) & & \\ f_2(\mathbf{z},k) = f(u,\mathbf{z},k) & - & f(u,\mathbf{z},l) & + & f_2(\mathbf{z},l) \\ f_2(\mathbf{z},k) = f(v,\mathbf{z},k) & - & f(v,\mathbf{z},l) & + & f_2(\mathbf{z},l) \end{cases}$$

$$f(u,\mathbf{z},k) - f(u,\mathbf{z},l) = f(v,\mathbf{z},k) - f(v,\mathbf{z},l)$$

Testing Pairwise Decomposability (2/3)

A cost function $f(X, \mathbf{Z}, Y)$ is pairwise decomposable w.r.t. X, Y iff $\forall \mathbf{z} \in D_{\mathbf{Z}}, \forall k, l \in D_Y (k < l)\ \forall u, v \in D_{\mathbf{X}}$:

$$f(u, \mathbf{z}, k) - f(u, \mathbf{z}, l) = f(v, \mathbf{z}, k) - f(v, \mathbf{z}, l)$$

A	B	C	f
0	0	0	1
0	0	1	0
0	1	0	4
0	1	1	1
1	0	0	3
1	0	1	2
1	1	0	7
1	1	1	4

f pairwise decompose w.r.t A,C ?

$$f(000) - f(001) \stackrel{?}{=} f(100) - f(101)$$
$$1 - 0 \stackrel{?}{=} 3 - 2$$
$$f(010) - f(011) \stackrel{?}{=} f(110) - f(111)$$
$$4 - 1 \stackrel{?}{=} 7 - 4$$

Testing Pairwise Decomposability (2/3)

A cost function $f(X, \mathbf{Z}, Y)$ is pairwise decomposable w.r.t. X, Y iff $\forall \mathbf{z} \in D_{\mathbf{Z}}, \forall k, l \in D_Y (k < l)\ \forall u, v \in D_{\mathbf{X}}$:

$$f(u, \mathbf{z}, k) - f(u, \mathbf{z}, l) = f(v, \mathbf{z}, k) - f(v, \mathbf{z}, l)$$

A	B	C	f
0	0	0	1
0	0	1	0
0	1	0	4
0	1	1	1
1	0	0	3
1	0	1	2
1	1	0	7
1	1	1	4

f pairwise decompose w.r.t A,C ?

$$f(000) - f(001) \stackrel{?}{=} f(100) - f(101)$$

$$1 - 0 \stackrel{?}{=} 3 - 2 \quad \checkmark$$

$$f(010) - f(011) \stackrel{?}{=} f(110) - f(111)$$

$$4 - 1 \stackrel{?}{=} 7 - 4 \quad \checkmark$$

Testing Pairwise Decomposability (2/3)

A cost function $f(X, \mathbf{Z}, Y)$ is pairwise decomposable w.r.t. X, Y iff
$\forall \mathbf{z} \in D_{\mathbf{Z}}, \forall k, l \in D_Y (k < l)\ \forall u, v \in D_{\mathbf{X}}$:

$$f(u, \mathbf{z}, k) - f(u, \mathbf{z}, l) = f(v, \mathbf{z}, k) - f(v, \mathbf{z}, l)$$

A	B	C	f
0	0	0	1
0	0	1	0
0	1	0	4
0	1	1	1
1	0	0	3
1	0	1	2
1	1	0	7
1	1	1	4

A	B	f_1
0	0	0
0	1	1
1	0	2
1	1	4

$+$

B	C	f_2
0	0	1
0	1	0
1	0	3
1	1	0

Pairwise Decomposability : dealing with determinism (3/3)

A cost function $f(X, \mathbf{Z}, Y)$ is pairwise decomposable w.r.t. X, Y iff $\forall \mathbf{z} \in D_{\mathbf{Z}}, \forall k, l \in D_Y (k < l)\ \forall u, v \in D_{\mathbf{X}}$:

$$f(u, \mathbf{z}, k) \boxminus f(u, \mathbf{z}, l) \circeq f(v, \mathbf{z}, k) \boxminus f(v, \mathbf{z}, l)$$

A	B	C	f
0	0	0	5
0	0	1	$\top$
0	1	0	6
0	1	1	5
0	2	0	$\top$
0	2	1	$\top$
1	0	0	4
1	0	1	$\top$
1	1	0	2
1	1	1	1
1	2	0	3
1	2	1	1

pairwise decomposable w.r.t A,C ?

$$\begin{array}{ccc} f(000) \boxminus f(001) & \circeq & f(100) \boxminus f(101) \\ 5 \boxminus \top & \circeq & 4 \boxminus \top \\ f(010) - f(011) & \circeq & f(110) - f(111) \\ 6 \boxminus 5 & \circeq & 2 \boxminus 1 \\ f(020) - f(021) & \circeq & f(120) - f(121) \\ \top \boxminus \top & \circeq & 3 \boxminus 1 \end{array}$$

Pairwise Decomposability : dealing with determinism (3/3)

A cost function $f(X, \mathbf{Z}, Y)$ is pairwise decomposable w.r.t. X, Y iff $\forall \mathbf{z} \in D_{\mathbf{Z}}, \forall k, l \in D_Y (k < l)\ \forall u, v \in D_{\mathbf{X}}$:

$$f(u, \mathbf{z}, k) \boxminus f(u, \mathbf{z}, l) \doteq f(v, \mathbf{z}, k) \boxminus f(v, \mathbf{z}, l)$$

A	B	C	f
0	0	0	5
0	0	1	$\top$
0	1	0	6
0	1	1	5
0	2	0	$\top$
0	2	1	$\top$
1	0	0	4
1	0	1	$\top$
1	1	0	2
1	1	1	1
1	2	0	3
1	2	1	1

pairwise decomposable w.r.t A,C ?

$$f(000) \boxminus f(001) \doteq f(100) \boxminus f(101)$$
$$5 \boxminus \top \doteq 4 \boxminus \top \quad \checkmark$$
$$f(010) - f(011) \doteq f(110) - f(111)$$
$$6 \boxminus 5 \doteq 2 \boxminus 1 \quad \checkmark$$
$$f(020) - f(021) \doteq f(120) - f(121)$$
$$\top \boxminus \top \doteq 3 \boxminus 1 \quad \checkmark$$

Pairwise Decomposability : dealing with determinism (3/3)

A cost function $f(X, \mathbf{Z}, Y)$ is pairwise decomposable w.r.t. X, Y iff $\forall \mathbf{z} \in D_{\mathbf{Z}}, \forall k, l \in D_Y (k < l)\ \forall u, v \in D_{\mathbf{X}}$:

$$f(u, \mathbf{z}, k) \boxminus f(u, \mathbf{z}, l) \circeq f(v, \mathbf{z}, k) \boxminus f(v, \mathbf{z}, l)$$

A	B	C	f
0	0	0	5
0	0	1	$\top$
0	1	0	6
0	1	1	5
0	2	0	$\top$
0	2	1	$\top$
1	0	0	4
1	0	1	$\top$
1	1	0	2
1	1	1	1
1	2	0	3
1	2	1	1

A	B	f_1
0	0	5
0	1	5
0	2	$\top$
1	0	4
1	1	1
1	2	1

B	C	f_2
0	0	0
0	1	$\top$
1	0	1
1	1	0
2	0	2
2	1	0

How to build f_1 and f_2?

- $f_1(X, \mathbf{Z}) = f[X, Z]$
- $f_2(\mathbf{Z}, Y) = (f - f_1)[\mathbf{Z}, Y]$

A	B	C	f
0	0	0	1
0	0	1	0
0	1	0	4
0	1	1	1
1	0	0	3
1	0	1	2
1	1	0	7
1	1	1	4

=

A	B	f_1
0	0	0
0	1	1
1	0	2
1	1	4

B	C	f_2
0	0	1
0	1	0
1	0	3
1	1	0

How to build f_1 and f_2?

- $f_1(X, \mathbf{Z}) = f[X, Z]$
- $f_2(\mathbf{Z}, Y) = (f - f_1)[\mathbf{Z}, Y]$

A	B	C	f
0	0	0	1
0	0	1	0
0	1	0	3
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	3
1	1	1	0

=

A	B	f_1
0	0	0
0	1	1
1	0	2
1	1	4

B	C	f_2
0	0	1
0	1	0
1	0	3
1	1	0

How to build f_1 and f_2?

- $f_1(X, \mathbf{Z}) = f[X, Z]$
- $f_2(\mathbf{Z}, Y) = (f - f_1)[\mathbf{Z}, Y]$

A	B	C	f
0	0	0	5
0	0	1	$\top$
0	1	0	6
0	1	1	5
0	2	0	$\top$
0	2	1	$\top$
1	0	0	4
1	0	1	$\top$
1	1	0	2
1	1	1	1
1	2	0	3
1	2	1	1

=

A	B	f_1
0	0	5
0	1	5
0	2	$\top$
1	0	4
1	1	1
1	2	1

+

B	C	f_2
0	0	0
0	1	$\top$
1	0	1
1	1	0
2	0	2
2	1	0

New local WCSP property

Pairwise decomposed (dec)

A WCSP is *pairwise decomposed* if all its cost functions are non pairwise decomposable.

Complexities

e number of functions, r max arity, d max domain size

Time $O(er^3d^{r+1})$

Space $O(ed^r)$, linear in problem size

Pairwise decomposable

- pairwise independent functions $(X \perp\!\!\!\perp Y \mid Z)_f$

$$f(X, Z, Y) = f_1(X, Z) + f_2(Z, Y)$$

- linear functions

$$f(X, Y, Z) = f_1(X) + f_2(Y) + f_3(Z)$$

- constraint trees [Meiri et al, 90]

$$f(X, Y, Z, T) = f_1(X, Y) + f_2(Y, Z) + f_3(Z, T)$$

(function `allequal`)

Non pairwise decomposable

- decomposition with extra variables (functions `soft regular`, `noisy-or`)
- $f(X, Y, Z) = f_1(X, Y) + f_2(Y, Z) + f_3(Z, X)$ (function `soft alldiff`)

What can we do with a non decomposable function ? ($r \geqslant 3$)

X	Y	Z	f
1	1	1	5
1	1	2	3
1	2	1	5
1	2	2	$\top$
2	1	1	4
2	1	2	2
2	2	1	1
2	2	2	1

pairwise decomposable w.r.t X,Z ?

$$f(000) \boxminus f(001) \stackrel{\circ}{=} f(100) \boxminus f(101)$$
$$5 \boxminus \top \stackrel{\circ}{\neq} 4 \boxminus 2$$

pairwise decomposable w.r.t X,Y ?

$$f(000) \boxminus f(010) \stackrel{\circ}{=} f(100) \boxminus f(110)$$
$$5 \boxminus 5 \stackrel{\circ}{\neq} 4 \boxminus 1$$

pairwise decomposable w.r.t Z,Y ?

$$f(000) \boxminus f(010) \stackrel{\circ}{=} f(001) \boxminus f(011)$$
$$5 \boxminus 5 \stackrel{\circ}{\neq} 3 \boxminus 4$$

What can we do with a non decomposable function ? ($r \geqslant 3$)

Project&Substract on all pairs of variables (ps)

$\forall X, Y \; f_i(X, Y) = f[X, Y]$ and $f \leftarrow f - f_i$

X	Y	Z	f
1	1	1	5
1	1	2	3
1	2	1	5
1	2	2	$\top$
2	1	1	4
2	1	2	2
2	2	1	1
2	2	2	1

What can we do with a non decomposable function ? ($r \geqslant 3$)

Project&Substract on all pairs of variables (ps)

$\forall X, Y \; f_i(X, Y) = f[X, Y]$ and $f \leftarrow f - f_i$

$$f(X, Y, Z) = b_1(X, Y) + b_2(Y, Z) + f_3(X, Y, Z)$$

X	Y	Z	f
1	1	1	5
1	1	2	3
1	2	1	5
1	2	2	$\top$
2	1	1	4
2	1	2	2
2	2	1	1
2	2	2	1

=

X	Y	b_1
1	1	3
1	2	5
2	1	2
2	2	1

+

Y	Z	b_2
1	1	2
1	2	0
2	1	0
2	2	0

+

X	Y	Z	f_3
1	1	1	0
1	1	2	0
1	2	1	0
1	2	2	$\top$
2	1	1	0
2	1	2	0
2	2	1	0
2	2	2	0

Experimental settings

Preprocessing (in `toulbar2` solver)

- *i*-`bounded variable elimination` (VE(i))
- `local consistency`
- `pairwise decomposition` (dec)
- `project&substract` (ps)

Search

- DFBB+VE(i) (`toulbar2` solver)
  - local consistency
  - 2-bounded variable elimination
- AOBB (`aolibWCSP` solver)
  - AND/OR graph with static mini buckets (SMB(j)) [Marinescu and Dechter, AAAI 2006]

Number of solved instances in 1 hour

	i=2	i=3	i=4	i=5	i=6	i=7
linkage (22)						
DFBB-VE(i)	14	14	15	13	12	10
DFBB-VE(i)+dec	16	**19**	17	13	14	13
DFBB-VE(i)+ps	**17**	16	17	18	16	16
DFBB-VE(i)+dec+ps	16	18	**19**	**19**	**19**	**17**
grids (32)						
DFBB-VE(i)	28	28	25	24	17	18
DFBB-VE(i)+dec	29	29	29	28	28	31
DFBB-VE(i)+ps	28	28	28	23	25	24
DFBB-VE(i)+dec+ps	**31**	**30**	**31**	**31**	**32**	**32**

Problem	DFBB-VE(i)		DFBB-VE(i) +dec+ps		AOBB-C+SMB(j)+VE(i)			AOBB-C+SMB(j)+VE(i) +dec+ps
	time (s)	i	time (s)	i	j	i	time (s)	time (s)
Linkage								
ped7	4.04	2	1.18	4	20	4	-	131.40
ped9	-		3.36	6	20	6	2225.40	94.20
ped18	149.71	4	3.19	5	20	5	30.96	16.00
ped19	-		-		20	6	-	-
ped20	3.46	4	0.39	6	16	6	771.90	66.31
ped23	0.09	4	0.05	3	12	3	5.1	1.50
ped25	1207.97	4	0.65	6	20	6	47.72	7.31
ped30	543.20	2	5.44	5	20	5	20.42	3.34
ped34	1.13	2	0.36	5	20	5	-	25.00
ped37	0.21	5	0.11	4	10	4	264.92	138.07
ped39	16.68	4	0.24	5	18	5	14.88	2.57
ped41	-		302.05	4	20	4	-	1651.23
ped42	1.94	4	0.34	6	16	6	-	154.96
ped44	-		505.46	5	20	5	-	83.74
ped50	0.90	4	0.18	4	12	4	316.92	521.92
Grids								
90-24-1	0.07	3	0.04	3	18	3	2323.01	0.01
90-26-1	0.29	3	0.07	3	16	3	2821.66	0.18
90-30-1	5.50	2	0.26	4	18	4	-	1.60
90-34-1	328.42	2	0.48	5	20	5	-	0.15
90-38-1	-		1.96	6	20	6	-	0.25

Conclusion and perspectives

Conclusion

- Determinism favours pairwise decomposition
- Synergy between Variable Elimination, Pairwise Decomposition and Project&Substract

Perspectives

- Pairwise decomposition during search
- Approximated pairwise decomposition
- Use of pairwise decomposition for probabilistic inference

Approximated pairwise decomposition

Approximated pairwise decomposition

A *pairwise ε-decomposition* of a cost function $f(\mathbf{S})$ with respect to two variables $X, Y \in \mathbf{S}$ ($X \neq Y$) is defined by two cost functions $f_1(\mathbf{S} \setminus \{Y\})$ and $f_2(\mathbf{S} \setminus \{X\})$ and $\varepsilon \in [0,1]$ such that :

$$\frac{1}{1+\varepsilon} f(\mathbf{S}) \leqslant f_1(\mathbf{S} \setminus \{Y\}) + f_2(\mathbf{S} \setminus \{X\}) \leqslant (1+\varepsilon) f(\mathbf{S})$$

Remark Exist a pairwise decomposable cost function between $\frac{1}{1+\varepsilon} f(\mathbf{S})$ and $(1+\varepsilon) f(\mathbf{S})$.

An example with $\varepsilon = 0.1$

X	Y	f
0	0	503
0	1	869
1	0	420
1	1	827

$$\begin{array}{ccc} f(0,0) - f(0,1) & \stackrel{?}{=} & f(1,0) - f(1,1) \\ 503 - 869 & \neq & 420 - 827 \\ -366 & & -407 \end{array}$$

X	$f_1(X)$
0	370
1	282

Y	$f_2(Y)$
0	138
1	470

X	Y	$\frac{1}{1+\varepsilon} f$	$\leqslant$	$f_1 + f_2$	$\leqslant$	$(1+\varepsilon) f$
0	0	457	$\leqslant$	508	$\leqslant$	553
0	1	790	$\leqslant$	840	$\leqslant$	955
1	0	381	$\leqslant$	420	$\leqslant$	462
1	1	751	$\leqslant$	752	$\leqslant$	909

Property

If a cost function $f(X, \mathbf{Z}, Y)$ is pairwise ε-decomposable w.r.t X,Y then $\forall z \in D_{\mathbf{Z}}$, $\forall k, l \in_Y$, $\forall u, v \in D_X$:

$$\frac{1}{1+\varepsilon} f(u,z,k) \boxminus (1+\varepsilon) f(u,z,l) \leqslant (1+\varepsilon) f(u,z,k) \boxminus \frac{1}{1+\varepsilon} f(u,z,l)$$

In other words :
$\forall z \in D_{\mathbf{Z}}$, $\forall k, l \in_Y$:

$$\max_{u \in D_X} \frac{1}{1+\varepsilon} f(u,z,k) \boxminus (1+\varepsilon) f(u,z,l) \leqslant \min_{u \in D_X} (1+\varepsilon) f(u,z,k) \boxminus \frac{1}{1+\varepsilon} f(u,z,l)$$