Maximum pseudo-likelihood estimator for nearest-neighbours Gibbs point processes

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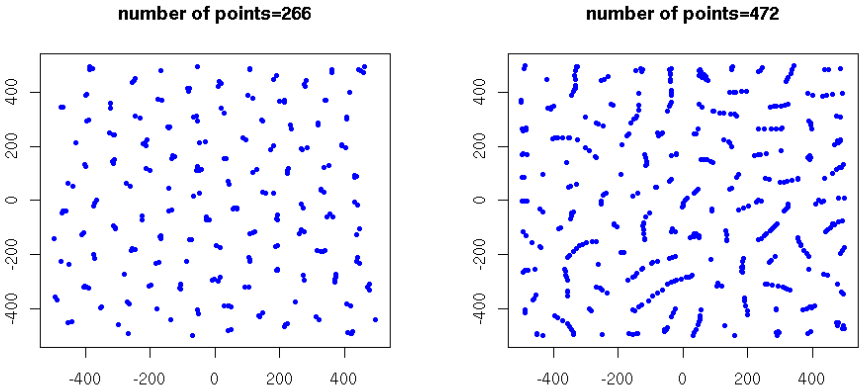
Objective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400
200
0
-200
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-200
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Outline

Outline

1. Gibbs point processes
   - Basic definition
   - Existence conditions based on the energy function
   - Description of some Gibbs models
2. Statistical model and inference method
3. Asymptotic results
   - Consistency of the mple estimator
   - Asymptotic normality of the mple estimator
4. Description of some examples and short simulation

Point processes: definition and notation

Notation

- $\mathcal{B}_b$: set of bounded borelian of $\mathbb{R}^d$.
- Ω_f, Ω, Ω_Λ: set of finite configurations of $\mathbb{R}^d$, set of configurations in $\mathbb{R}^d$, set of configurations in $\Lambda \subset \mathbb{R}^d$:
- Let $\Lambda \subset \mathbb{R}^d$ and $\varphi \in \Omega$, $\varphi_\Lambda := \varphi \cap \Lambda \in \Omega_\Lambda$

Point process in some bounded $\Lambda \subset \mathbb{R}^d$

A point process in Λ is a random variable Φ_Λ with values in Ω_Λ equipped with the smallest σ-field which make measurable all the maps $i_\Delta : \varphi \in \Omega_\Lambda \to |\varphi_\Delta|$ with $\Delta \subset \Lambda \in \mathcal{B}_b$.

Useful notation

$$\oint_\Lambda d\varphi\, g(\varphi) := \sum_{n=0}^{+\infty} \frac{1}{n!} \int_\Lambda \cdots \int_\Lambda dx_1 \cdots dx_n\, g\left(\{x_1, \cdots, x_n\}\right)$$

i.e. $\oint_\Lambda d\varphi$ means the summation over all configuration φ in Λ.

Poisson point process and Gibbs point process

We define:

$\rightarrow$ a poisson point process with intensity 1 in Λ with probability measure Q_Λ

$$Q_\Lambda(F) = \frac{1}{\exp(|\Lambda|)} \sum_{n=0}^{+\infty} \frac{1}{n!} \int_\Lambda \cdots \int_\Lambda dx_1 \cdots dx_n \mathbf{1}_F\left(\{x_1, \cdots, x_n\}\right)$$

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Poisson point process and Gibbs point process

We define:

→ a Gibbs point process in Λ with probability measure P_Λ

$$P_\Lambda(F) = \frac{1}{Z_\Lambda} \oint_\Lambda d\varphi \mathbf{1}_F(\varphi) \exp\left(-V(\varphi)\right)$$

where $Z_\Lambda < +\infty$ as soon as $V(\varphi) > -K|\varphi|$ (i.e. $V(\cdot)$ is **stable**).

Useful notation

$$\oint_\Lambda d\varphi\, g(\varphi) := \sum_{n=0}^{+\infty} \frac{1}{n!} \int_\Lambda \cdots \int_\Lambda dx_1 \cdots dx_n\; g\left(\{x_1, \cdots, x_n\}\right)$$

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Poisson point process and Gibbs point process

We define:

$\rightarrow$ a Gibbs point process in $\mathrm{I\!R}^d$ with conditional probability measure $P_\Lambda(\cdot|\varphi^o)$ for all φ^o

$$P_\Lambda(F|\varphi^o) = \frac{1}{Z_\Lambda(\varphi^o)} \oint_\Lambda d\varphi \mathbf{1}_F(\varphi) \exp\left(-V(\varphi|\varphi^o_{\Lambda^c})\right)$$

where $V(\varphi|\varphi^o_{\Lambda^c}) := V(\varphi \cup \varphi^o_{\Lambda^c}) - V(\varphi^o_{\Lambda^c})$ is the energy required to insert the points of φ in $\varphi^o_{\Lambda^c}$

Framework (of the presentation)

Restricted to stationary Gibbs point processes based on energy function related to some graph $G_2(\varphi)$:

$$V(\varphi) = \sum_{k=1}^{K_{max}} \left\{ \sum_{\xi \in G_k(\varphi)} u^{(k)}(\xi;\varphi) \right\}$$

with $G_k(\varphi)$: set of cliques of order k of φ

satisfying the following Assumptions

E1 $V(\cdot)$ is invariant by translation.

E2 Locality of the local energy: $\exists D > 0$ such that

$$V(0|\varphi) = V(0|\varphi \cap \mathcal{B}(0,D)).$$

(can be replaced by a quasi-locality assumption).

E3 Stability of the local energy: $\exists K \geq 0$ such that

$$V(0|\varphi) \geq -K.$$

Framework (of the presentation)

Restricted to stationary Gibbs point processes based on energy function related to some graph $G_2(\varphi)$:

$$V(\varphi) = \theta\,|\varphi| + \sum_{\xi \in G_2(\varphi)} u(\xi;\varphi), \qquad \theta \in \mathbb{R}$$

$\Longrightarrow$ pairwise interaction point processes.

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This framework includes

- models based on the usual complete graph $G(\varphi) = \mathcal{P}_2(\varphi)$ with pairwise interaction function satisfying a hard-core or inhibition condition and with finite range.
- models based on the (slightly modified) Delaunay graph $G_2(\varphi) = Del_{2,\beta}^{\beta_0}(\varphi)$ with pairwise interaction function bounded and with finite range.

▸ Delaunay graph

Definition of $Del_{2,\beta}^{\beta_0}(\varphi)$

Let $Del_3(\varphi)$ denote the "Delaunay triangles", let $\beta_0 \in [0, \pi/3[$ and let $\beta(\psi)$ denote the smallest angle of some triangle ψ. Then,

$$Del_{3,\beta}^{\beta_0}(\varphi) = \{\psi \in Del_3(\psi), \beta(\psi) \geq \beta_0\} \text{ and } Del_{2,\beta}^{\beta_0} = \cup_{\psi \in Del_{3,\beta}^{\beta_0}} \mathcal{P}_2(\psi)$$

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Model parametrization (1)

Parametrization of the function $u(\cdot)$

- Let $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ where $\boldsymbol{\Theta}$ is a compact of $\mathbb{R}^{p+1}$.
- Energy function described by:

$$V(\varphi;\boldsymbol{\theta}) = \theta_1|\varphi| + \sum_{\xi \in G_2(\varphi)} u(||\xi||;\boldsymbol{\theta})$$

Local energy: energy to insert x in some configuration φ

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$$V(x|\varphi; \boldsymbol{\theta}) = \theta_1 + \sum_{\xi \in G_2(\varphi \cup \{x\})} u(||\xi||; \boldsymbol{\theta}) - \sum_{\xi \in G_2(\varphi)} u(||\xi||; \boldsymbol{\theta}).$$

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$$V(x|\varphi; \boldsymbol{\theta}) = \theta_1 + \sum_{y \in \varphi} u(||y - x||; \boldsymbol{\theta}) \qquad \text{when } G_2(\varphi) = \mathcal{P}_2(\varphi).$$

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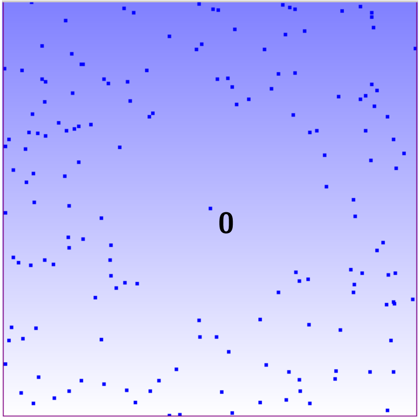
$$V(x|\varphi; \boldsymbol{\theta}) = \theta_1 + \underbrace{\sum_{\xi \in G_2(\varphi \cup x) \setminus G_2(\varphi)} u(||\xi||; \boldsymbol{\theta})}_{\textbf{positive contribution}} - \underbrace{\sum_{\xi \in G_2(\varphi) \setminus G_2(\varphi \cup \{x\})} u(||\xi||; \boldsymbol{\theta})}_{\textbf{negative contribution}},$$

for a general graph such as $G_2(\varphi) = Del_{2,\beta}^{\beta_0}(\varphi)$.

Inhibition Delaunay Gibbs Process

Local energy:
$V(\mathbf{0}|\varphi) := V(\mathbf{0} \cup \varphi) - V(\varphi)$



$$V_2^+(\mathbf{0}|\varphi) = \sum_{\substack{\xi^+ \in G(\mathbf{0} \cup \varphi) \\ \xi^+ \notin G(\varphi)}} u\left(||\xi^+||; \boldsymbol{\theta}\right)$$

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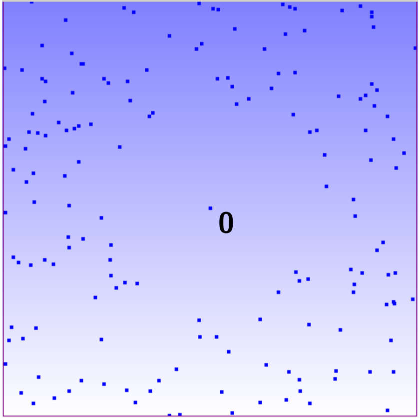

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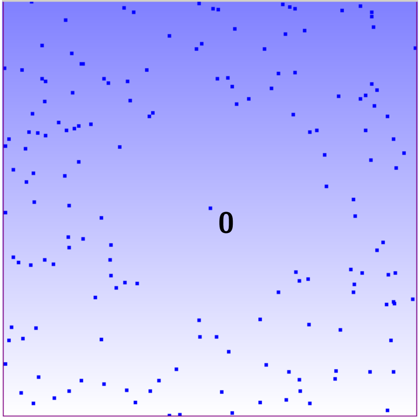

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when $G(\varphi) \subset G(\mathbf{0} \cup \varphi)$

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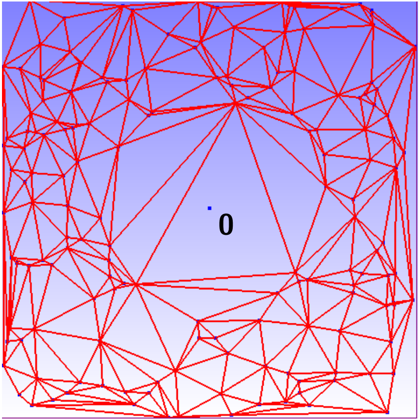

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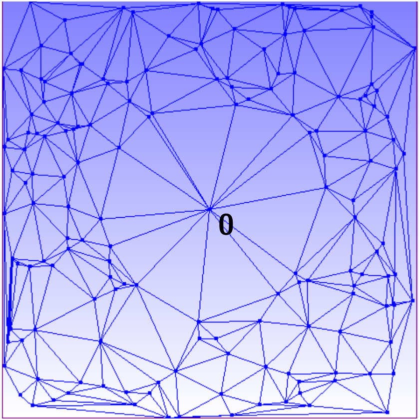

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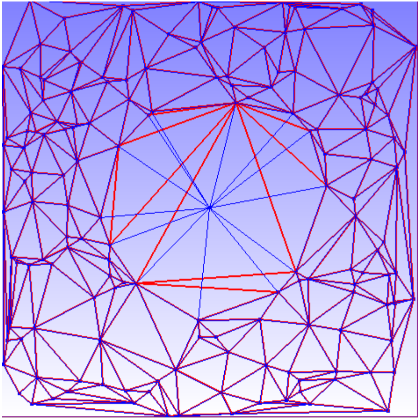

Model parametrization (2)

Particular case : exponential family

$$u(||\xi||;\boldsymbol{\theta}) = \sum_{i=2}^{p+1} \theta_i u_i(||\xi||).$$

$\Longrightarrow$ Exponential energy function: $\boxed{V(\varphi;\boldsymbol{\theta}) = \boldsymbol{\theta}^T \mathbf{u}(\varphi),}$ where $\mathbf{u}(\cdot) = (u_1(\cdot), \ldots, u_{p+1}(\cdot))$ with

$$u_1(\varphi) = |\varphi| \quad \text{and} \quad u_i(\varphi) = \sum_{\xi \in G_2(\varphi)} u_i(||\xi||).$$

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Main example: multi-Strauss pairwise interaction function

Let $d_1 < d_2 < \ldots < d_{p+1}$ some **fixed** realnumbers. Define for $i = 2, \ldots, p+1$,

$$u_i(||\xi||) = \mathbf{1}\left(||\xi|| \in]d_{i-1}, d_i]\right).$$

Thus, for our two Gibbs models

$$V(\varphi; \boldsymbol{\theta}) = \theta_1 |\varphi| + \sum_{i=2}^{p+1} \theta_i u_i(\varphi),$$

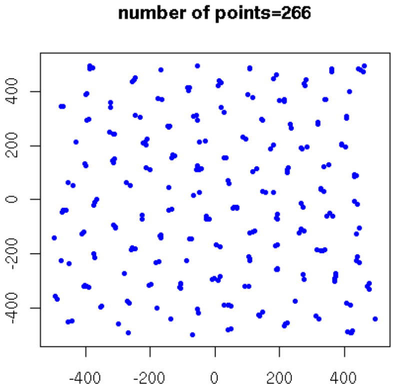
where $u_i(\varphi)$ is interpreted as

- (when $G_2(\varphi) = \mathcal{P}_2(\varphi)$) the number of points of φ in the class of distances $]d_{i-1}, d_i]$.
- (when $G_2(\varphi) = Del_{2,\beta}^{\beta_0}(\varphi)$) the number of Delaunay edges of φ in the class of distances $]d_{i-1}, d_i]$.

An example

$$\boldsymbol{\theta} = (1, 2, 4),\ \mathbf{d} = (0, 20, 80)$$

$G_2(\varphi) = \mathcal{P}_2(\varphi)$



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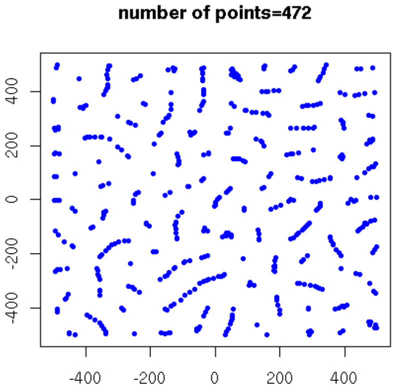

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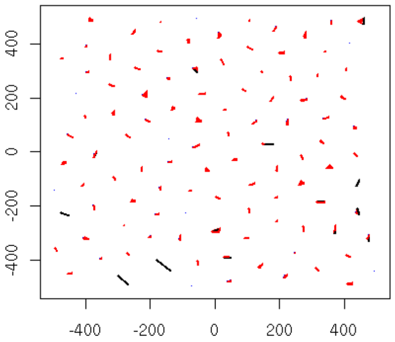
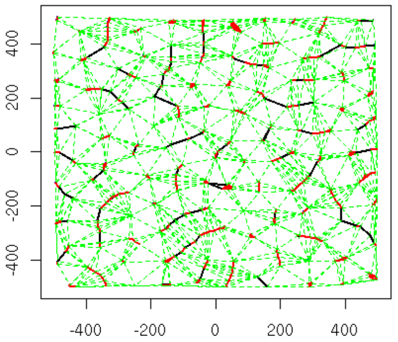
Small 213 (0.6%), Medium 15 (0%), Large 35017 (99.4%)



$G_2(\varphi) = Del_{2,\beta}^{\beta_0}(\varphi)$

Small 415 (30.3%), Medium 64 (4.7%), Large 891 (65%)



Inference method

Data

- Realization of a p.p. with energy function $V(\cdot;\boldsymbol{\theta}^{\star})$ in some domain $\Lambda \subset \mathbb{R}^d$ satisfying Assumptions $\mathbf{E_1}$ to $\mathbf{E_3}$.
- $\boldsymbol{\theta}^{\star}$ true parameter to be estimated, $P_{\boldsymbol{\theta}^{\star}}$ associated Gibbs measure.

Usual parametric methods

- maximum likelihood estimator: drawback= computation of the normalizing constant.
- maximum pseudo-likelihood estimator (Besag (1968), Jensen and Møller (1991), ...)
- Takacs-Fiksel estimator (based on the refined Campbell theorem): competitive with respect to the MPLE.

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- Pseudo-likelihood function (Jensen and Møller (1991))

$$PL_\Lambda(\varphi;\boldsymbol{\theta}) = \exp\left(-\int_\Lambda \exp\left(-V\left(x|\varphi;\boldsymbol{\theta}\right)\right)dx\right)\prod_{x\in\varphi_\Lambda}\exp\left(-V\left(x|\varphi\setminus x;\boldsymbol{\theta}\right)\right).$$

- Log-pseudo-likelihood function

$$LPL_\Lambda(\varphi;\boldsymbol{\theta}) = -\int_\Lambda \exp\left(-V\left(x|\varphi;\boldsymbol{\theta}\right)\right)dx - \sum_{x\in\varphi_\Lambda} V\left(x|\varphi\setminus x;\boldsymbol{\theta}\right)$$

Different contributions to asympt. results (when $G_2(\varphi) = \mathcal{P}_2(\varphi)$)

- Jensen and Møller (1991): consistency of the MPLE, exponential family.
- Jensen and Kunsch (1994): asymptotic normality of the MPLE, exponential family $\boldsymbol{\theta} = (z, \beta)$
- Mase (1995, 1999): consistency and asymptotic normality, $\boldsymbol{\theta} = (z, \beta)$ Ruelle's class of superstable pairwise interaction p.p.

Outline

Asymptotic results - Introduction

Definition of the estimator

- Define

$$U_n(\boldsymbol{\theta}) = -\frac{1}{|\Lambda_n|} LPL_{\Lambda_n}(\varphi; \boldsymbol{\theta})$$

- Maximum pseudo-likelihood estimator:

$$\widehat{\boldsymbol{\theta}}_n(\varphi) = \mathrm{argmax}_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \, LPL_{\Lambda_n}(\varphi; \boldsymbol{\theta}) = \mathrm{argmin}_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \, U_n(\boldsymbol{\theta})$$

Lemma

Under certain Assumptions, $U_n(\cdot)$ defines a contrast function: there exists a function $K(\cdot, \theta^\star)$ such that $P_{\theta^\star}$-a.s. $U_n(\theta) - U_n(\theta^\star) \to K(\theta, \theta^\star)$, where $K(\cdot, \theta^\star)$ is a positive function and is zero if and only if $\theta = \theta^\star$.

$\Longrightarrow$ results on minimum contrast estimators (Guyon (1992))

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$$\widehat{\boldsymbol{\theta}}_n(\varphi) = \operatorname{argmax}_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} LPL_{\Lambda_n}(\varphi; \boldsymbol{\theta}) = \operatorname{argmin}_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} U_n(\boldsymbol{\theta})$$

Lemma

Under certain Assumptions, $U_n(\cdot)$ defines a contrast function: there exists a function $K(\cdot, \boldsymbol{\theta}^\star)$ such that $P_{\boldsymbol{\theta}^\star}$-a.s. $U_n(\boldsymbol{\theta}) - U_n(\boldsymbol{\theta}^\star) \to K(\boldsymbol{\theta}, \boldsymbol{\theta}^\star)$, where $K(\cdot, \boldsymbol{\theta}^\star)$ is a positive function and is zero if and only if $\boldsymbol{\theta} = \boldsymbol{\theta}^\star$.

$\Longrightarrow$ results on minimum contrast estimators (Guyon (1992))

Consistency of the MPLE: assumptions - general case

C$_1$ $(\Lambda_n)_{n\geq 1}$ is a regular sequence of domains such that $\Lambda_n \longrightarrow \mathbb{R}^2$ as $n \longrightarrow +\infty$.

C$_2$ For all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$,

$$V\left(0|\cdot;\boldsymbol{\theta}\right) \in L^1(P_{\boldsymbol{\theta}^\star}).$$

C$_3$ For all $\boldsymbol{\theta} \in \boldsymbol{\Theta} \setminus \boldsymbol{\theta}^\star$,

$$P_{\boldsymbol{\theta}^\star}\Big(\{\varphi, \quad V\left(0|\varphi;\boldsymbol{\theta}\right) \neq V\left(0|\varphi;\boldsymbol{\theta}^\star\right)\} \Big) > 0$$

C$_4$ For all $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \boldsymbol{\Theta}$, there exists $c > 0$ such that $P_{\boldsymbol{\theta}^\star}-$a.s.

$$\left|V\left(0|\Phi;\boldsymbol{\theta}\right) - V\left(0|\Phi;\boldsymbol{\theta}'\right)\right| \leq ||\boldsymbol{\theta} - \boldsymbol{\theta}'||^c g(0,\Phi)$$

where $g(\cdot,\cdot)$ is a function such that for all x, $g(0,\Phi) = g(x,\Phi_x)$ and such that $g(0,\cdot) \in L^1(P_{\boldsymbol{\theta}^\star})$.

Consistency of the MPLE: assumptions - exponential case

Conditions $\mathbf{C_2}$ and $\mathbf{C_4}$ (resp. $\mathbf{C_3}$) can be replaced by $\mathbf{C_{2,4}^{exp}}$ (resp. $\mathbf{C_3^{exp}}$) where

$\mathbf{C_{2,4}^{exp}}$ There exists $\varepsilon > 0$ such that for all $i = 1, \ldots, p+1$

$$u_i(0|\cdot) \in L^{1+\varepsilon}(P_{\boldsymbol{\theta}^\star}).$$

$\mathbf{C_3^{exp}}$ **Identifiability condition** : There exists $A_1, \ldots, A_{p+1}$, $p+1$ disjoint events of Ω such that $P_{\boldsymbol{\theta}^\star}(A_i) > 0$ and such that for all $\varphi_1, \ldots, \varphi_{p+1} \in A_1 \times \cdots \times A_{p+1}$ the $(p+1) \times (p+1)$ matrix with entries $u_j(0|\varphi_i)$ is constant and invertible.

Consistency: statement of the result

Proposition (consistency)

Assume $P_{\boldsymbol{\theta}^\star}$ stationary, then under Assumptions $\mathbf{C_1}$ to $\mathbf{C_4}$ in the general case or under Assumptions $\mathbf{C_1}$, $\mathbf{C_{2,4}^{exp}}$ and $\mathbf{C_3^{exp}}$ in the exponential case, we have $P_{\boldsymbol{\theta}^\star}-$almost surely, as $n \to +\infty$,

$$\widehat{\boldsymbol{\theta}}_n(\Phi) \to \boldsymbol{\theta}^\star$$

Tools

- Glötz Theorem, refined Campbell theorem.
- General ergodic theorems obtained by Nguyen and Zessin (1979).
- General result concerning the consistency of minimum contrast estimators obtained by Guyon (1992).

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Asymptotic normality: assumptions - general case (1)

N_1 The point process is observed in a domain $\Lambda_n \oplus D = \cup_{x\in\Lambda_n}\mathcal{B}(x,D)$, where $\Lambda_n \subset \mathbb{R}^2$ can de decomposed into $\cup_{i\in I_n}\Lambda_{(i)}$ where for $i=(i_1,i_2)$

$$\Lambda_{(i)} = \left\{ z \in \mathbb{R}^2, \widetilde{D}\left(i_j - \frac{1}{2}\right) \leq z_j \leq \widetilde{D}\left(i_j - \frac{1}{2}\right), j = 1,2 \right\}$$

for some $\widetilde{D} > 0$. As $n \to +\infty$, we also assume that $\Lambda_n \to \mathbb{R}^2$ such that $|\Lambda_n| \to +\infty$ and $\frac{|\partial\Lambda_n|}{|\Lambda_n|} \to 0$

N_2 $V(0|\cdot;\boldsymbol{\theta})$ is twice times differentiable in $\boldsymbol{\theta} = \boldsymbol{\theta}^\star$ and for all $j,k = 1,\ldots,p+1$, there exists $\varepsilon > 0$ such that the variables

$$\frac{\partial V}{\partial \theta_j}(0|\cdot;\boldsymbol{\theta}^\star)^{3+\varepsilon} \quad \text{and} \quad \frac{\partial^2 V}{\partial\theta_j\partial\theta_k}(0|\cdot;\boldsymbol{\theta}^\star) \in L^1(P_{\boldsymbol{\theta}^\star})$$

Asymptotic normality: assumptions - general case (2)

N₃ The matrix

$$\underline{\boldsymbol{\Sigma}}(\widetilde{D},\boldsymbol{\theta}^\star) = \widetilde{D}^{-2} \sum_{|i|\leq\left[\frac{D}{\widetilde{D}}\right]+1} \mathbf{E}_{\boldsymbol{\theta}^\star}\left(\mathbf{LPL}^{(1)}_{\Lambda_0}(\Phi;\boldsymbol{\theta}^\star)\,\mathbf{LPL}^{(1)}_{\Lambda_i}(\Phi;\boldsymbol{\theta}^\star)^T\right)$$

is symmetric and definite positive.

The vector $\mathbf{LPL}^{(1)}_{\Lambda_i}(\varphi;\boldsymbol{\theta})$ is defined for $j=1,\ldots,p+1$ by

$$\left(\mathbf{LPL}^{(1)}_{\Lambda_i}(\varphi;\boldsymbol{\theta})\right)_j = \int_{\Lambda_{(i)}} \frac{\partial V}{\partial\theta_j}(x|\varphi;\boldsymbol{\theta})\exp\left(-V\left(x|\varphi;\boldsymbol{\theta}\right)\right)dx - \sum_{x\in\varphi_{\Lambda_{(i)}}} \frac{\partial V}{\partial\theta_j}(x|\varphi\setminus x;\boldsymbol{\theta}).$$

N₄ $\forall \mathbf{y}\in \mathbb{R}^{p+1}\setminus\{\mathbf{0}\}$

$$P_{\boldsymbol{\theta}^\star}\left(\left\{\varphi,\ \mathbf{y}^T\mathbf{V}^{(1)}(0|\varphi;\boldsymbol{\theta}^\star)\neq 0\right\}\right) > 0,$$

where for $i=1,\ldots,p+1$, $(\mathbf{V}^{(1)}(0|\varphi;\boldsymbol{\theta}^\star))_i = \frac{\partial V}{\partial\theta_i}(0|\varphi;\boldsymbol{\theta}^\star)$.

Asymptotic normality: assumptions - general case (3)

N$_5$ There exists a neighborhood $\mathcal{W}$ of $\boldsymbol{\theta}^\star$ such that $V(\cdot;\boldsymbol{\theta})$ is twice times continuously differentiable for all $j,k=1,\ldots,p+1$, we have

$$\left|\frac{\partial V}{\partial\theta_j}(0|\Phi;\boldsymbol{\theta})-\frac{\partial V}{\partial\theta_j}(0|\Phi;\boldsymbol{\theta}^\star)\right|\leq||\boldsymbol{\theta}-\boldsymbol{\theta}^\star||^{c_1}\;h_1(0,\Phi),$$

and

$$\left|\frac{\partial^2 V}{\partial\theta_j\partial\theta_k}(0|\Phi;\boldsymbol{\theta})-\frac{\partial^2 V}{\partial\theta_j\partial\theta_k}(0|\Phi;\boldsymbol{\theta}^\star)\right|\leq||\boldsymbol{\theta}-\boldsymbol{\theta}^\star||^{c_2}\;h_2(0,\Phi),$$

with $c_1,c_2>0$ and $h_1(\cdot,\cdot),h_2(\cdot,\cdot)$ two functions such that, for all x, $h_i(0,\Phi)=h_i(x,\Phi_x)$ and such that $h_1(0,\cdot)^2$ and $h_2(0,\cdot)\in L^1(P_{\boldsymbol{\theta}^\star})$.

Asymptotic normality: assumptions - exponential case

Assumptions $\mathbf{N_2}$ and $\mathbf{N_5}$ (resp. $\mathbf{N_4}$) can be replaced by $\mathbf{N_{2,5}^{exp}}$ (resp. $\mathbf{N_4^{exp}}$) where

$\mathbf{N_{2,5}^{exp}}$ For $i = 1, \ldots, p+1$, there exists $\varepsilon > 0$ such that $u_i(0|\cdot) \in L^{3+\varepsilon}(P_{\boldsymbol{\theta}^\star})$.

$\mathbf{N_4^{exp}}$ $= \mathbf{C_3^{exp}}$

Asymptotic normality: statement of the result

Proposition (asymptotic normality)

Assume $P_{\theta^\star}$ stationary, then under Assumptions $\mathbf{N_1}$ to $\mathbf{N_5}$ in the general case or under Assumptions $\mathbf{N_1}$ $\mathbf{N_{2,5}^{exp}}$ $\mathbf{C_3^{exp}}$ and $\mathbf{N_3}$ in the exponential case, we have, for any fixed $\widetilde{D}$ fixed

$$|\Lambda_n|^{1/2}\, \widehat{\underline{\boldsymbol{\Sigma}}}_n(\widetilde{D},\widehat{\boldsymbol{\theta}}_n)^{-1/2}\, \underline{\mathbf{U_n}}^{(2)}(\widehat{\boldsymbol{\theta}}_n)\, \left(\widehat{\boldsymbol{\theta}}_n-\boldsymbol{\theta}^\star\right) \to \mathcal{N}\left(0,\underline{\mathbf{I}}_{p+1}\right),$$

where for some $\boldsymbol{\theta}$ and some finite configuration φ

$$\widehat{\underline{\boldsymbol{\Sigma}}}_n(\widetilde{D},\boldsymbol{\theta}) = |\Lambda_n|^{-1}\widetilde{D}^{-2}\sum_{i\in I_n}\ \sum_{|j-i|\leq\left[\frac{D}{\widetilde{D}}\right]+1, j\in I_n} \mathbf{LPL}^{(1)}_{\Lambda_i}(\varphi;\boldsymbol{\theta})\, \mathbf{LPL}^{(1)}_{\Lambda_j}(\varphi;\boldsymbol{\theta})^T$$

Tools

- Asympt. normality for minimum contrast estimators (Guyon (1992)).
- Central Limit Theorem obtained by Jensen and Künsch (1994).

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Tools

- Asympt. normality for minimum contrast estimators (Guyon (1992)).
- Central Limit Theorem obtained by Jensen and Künsch (1994).

Outline

A useful corollary

A particular class of exponential family

M There exists $K_1, K_2 > 0$ such that for any finite configuration φ, we have for all x

$$-K_1 \leq u_i(x|\varphi) \leq K_2, \qquad \text{for } i = 1, \ldots, p+1.$$

Assumption $\mathbf{M} \Longrightarrow \mathbf{C}_{2,4}^{\mathbf{exp}}$ and $\mathbf{N}_{2,5}^{\mathbf{exp}}$.

Corollary

Assume $P_{\theta^\star}$ stationary, then under Assumption $\mathbf{M}$ and $\mathbf{C}_{3}^{\mathbf{exp}}$, the consistency is valid. And in addition under Assumption $\mathbf{N_3}$ the asymptotic normality is ensured.

Back to the multi-Strauss pairwise interaction p.p.

$$V(\varphi;\boldsymbol{\theta}) = \theta_1|\varphi| + \sum_{i=2}^{p+1}\theta_i \sum_{\xi\in Del_{2,\beta}^{\beta_0}(\varphi)} \mathbf{1}(||\xi|| \in]d_{i-1}, d_i]).$$

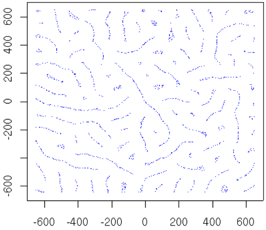
Assumption $\mathbf{M}$, $\mathbf{C_3^{exp}}$ and $\mathbf{N_3}$

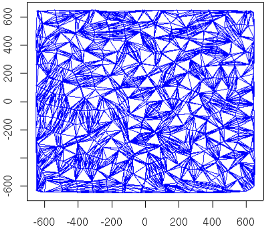
- Assumption $\mathbf{M}$: proved in Bertin, Billiot and Drouilhet (1999).
- Assumption $\mathbf{C_3^{exp}}$: verified by considering particular sets of configurations of two points in a domain $\Delta = \{z \in \mathbb{R}^2, -D \leq z_i \leq D, i = 1, 2\}$.
- Assumption $\mathbf{N_3}$: verified for this model by using an inequality obtained by Jensen and Künsch and then by considering particular sets of configurations of tree points in $\cup_{|i|\leq 1}\Lambda_{(i)}$.

Short simulation study

Parameters

- $\boldsymbol{\theta}^\star = (0, 2, 4)$, $\mathbf{d} = (0, 20, 80)$
- $m = 5000$ replications generated in the domain $[-600, 600]^2$.





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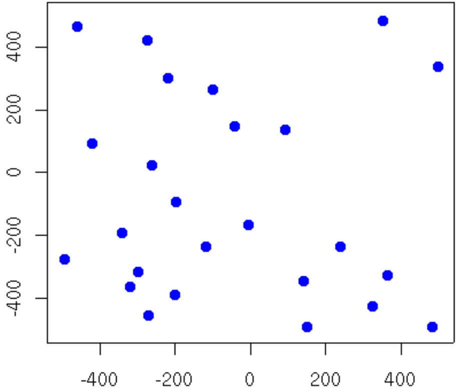
	Estimations of $\theta_2^\star$		Estimations of $\theta_3^\star$	
Domain Λ_n	Mean	(Std Dev.)	Mean of Estim.	(Std Dev.)
$[-250, 250]^2$	2.068	0.104	4.382	0.786
$[-350, 350]^2$	2.049	0.071	4.223	0.551
$[-450, 450]^2$	2.041	0.056	4.144	0.436

Perspectives

- concerning the multi-Strauss pairwise interaction point process based on the Delaunay graph: automatic estimation of the different d_i, $i = 1, \ldots, p+1$.
- A larger simulation study is needed:
  1. to compare models based on the Delaunay graph and the complete graph.
  2. to investigate other nearest-neighbour models, models based on cliques of order larger than 2, marked nearest-neighbour Gibbs point processes,...
- Nonparametric estimation of the pairwise interaction function for nearest-neighbour Gibbs point processes.

Delaunay graph, delaunay triangulation



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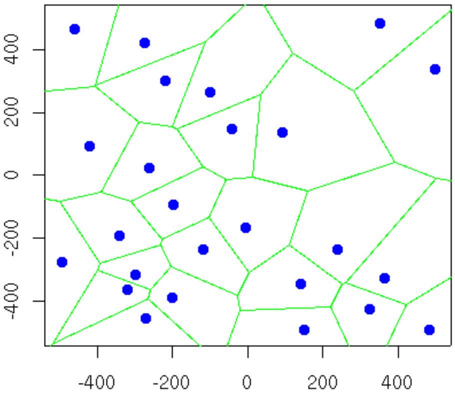
Delaunay graph, delaunay triangulation



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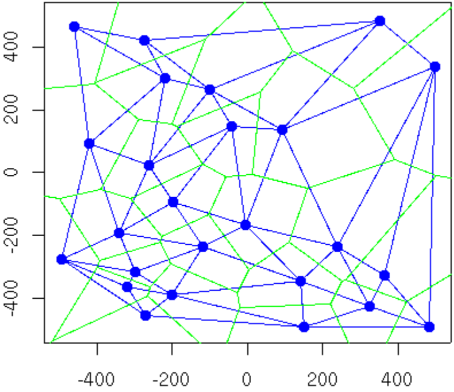
Delaunay graph, delaunay triangulation



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