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A new class of processes for formalizing individual-based models : the Semi-Semi-Markov Processes

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Origin : epidemiological problem

Study the propagation of a disease (BVD : Bovine Viral Diarrhoea) in a dairy herd
3 types of individual transitions :

- Branching (population dynamic due to calves births)
- Groups changes (sojourn time in a group depends on the physiological status and on age) :
  4 groups : Calves, Heifers before breeding, Heifers after breeding, Dairy cows
- Health status changes depending on the population health status
  Complex disease with two kinds of excreting animals : the Transient Infectives, Persistent Infective

$\Longrightarrow$ individual-based stochastic model with individual state-dependent semi-Markovian transition laws

Individual-based models

- To model population dynamics when individuals are marked by personal characteristics and/or history or when the next state-change is driven by complex rule decisions depending on the current state of the population
- To calculate empirical distributions at the scale of the population from simulated individual trajectories ("bottom-up approach")
- Litterature on individual-based models has considerably increased thanks to the increase of the computers capacity and the popularization of informatics
- No mathematical formalism : validation of this approach ? how is the process at the level of the population ?

Goal

To build a rigorous mathematical formalism of these models at the population level (top-down)

$\Longrightarrow$ good readability of the different model components independently of the programming language

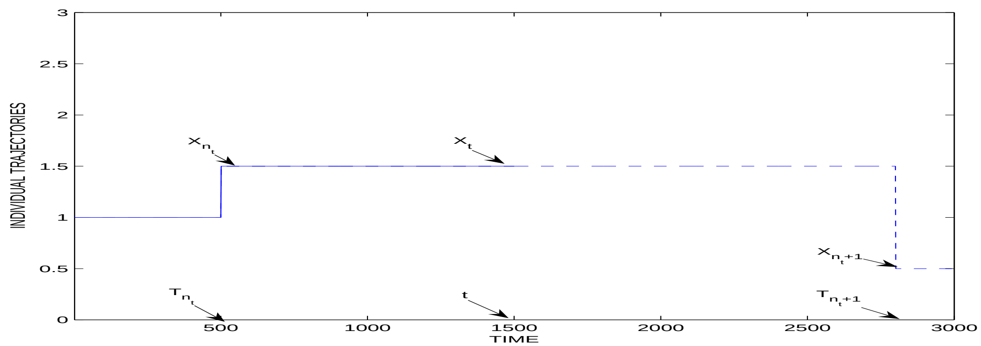
$\Longrightarrow$ validate the empirical distributions by analytical results

Outline

1. Homogeneous Semi-Markov Process (SMP) for one individual
   kernel, transition rates,simulation algorithm, probability law, asymptotic behavior
2. Homogeneous Semi-Semi-Markov Process for a closed population :
   kernel, simulation algorithm, transition rates, probability law
3. Conclusion

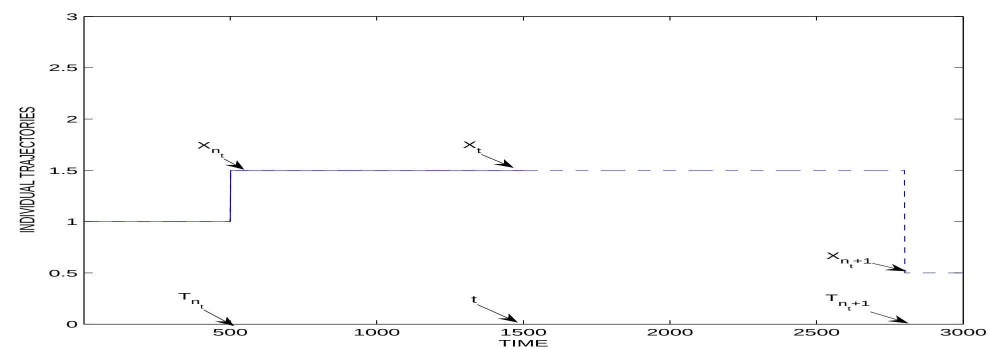
Homogeneous Semi-Markov Process (SMP) for one individual ω

Feller W. (1964), Çinlar (1975), Kulkarni, V. (1995), Becker G. *et al.* (1999), Iosifescu M. (1999)



Let $\{X_n(\omega), T_n(\omega)\}_n$ be an homogeneous Markov Renewal Process (MRP)
$\{X_t(\omega)\}_t$ is an homogeneous semi-Markov process if

$$X_t(\omega) \;=\; X_{n_t}(\omega) 1_{\{n_t(\omega)=\sup\{n:T_n(\omega)\leq t\}\}}$$



$\Delta T_{n+1} = T_{n+1} - T_n$ (waiting time between 2 jumps)
$\{X_n, T_n\}$ is a MRP if

$$\begin{aligned} P(X_{n+1} = j, \Delta T_{n+1} \le \tau | X_n, \ldots, X_0, T_n, \ldots, T_0) &= P(X_{n+1} = j, \Delta T_{n+1} \le \tau | X_n) \\ &\overset{homog.}{=} P(X_1 = j, \Delta T_1 \le \tau | X_0) \end{aligned}$$

Law of $\{X_t\}_t \iff$ law of $\{X_n, T_n\}_n \iff \{Q_{i,j}(.)\}_{i,j}$

Kernels (transition laws)

$$\begin{aligned} Q_{i,j}(\tau) &= P(\Delta T_1 \le \tau, X_1 = j | X_0 = i) \\ &= P(\Delta T_1 \le \tau | X_1 = j, X_0 = i) P(X_1 = j | X_0 = i) \\ &= F_{i,j}(\tau) P(i, j) \end{aligned}$$

$F_{i,j}(\tau)$: cdf of the sojourn time in i before jumping in j ; $P(i,j)$: transition probability of $\{X_n\}$

Transition rates

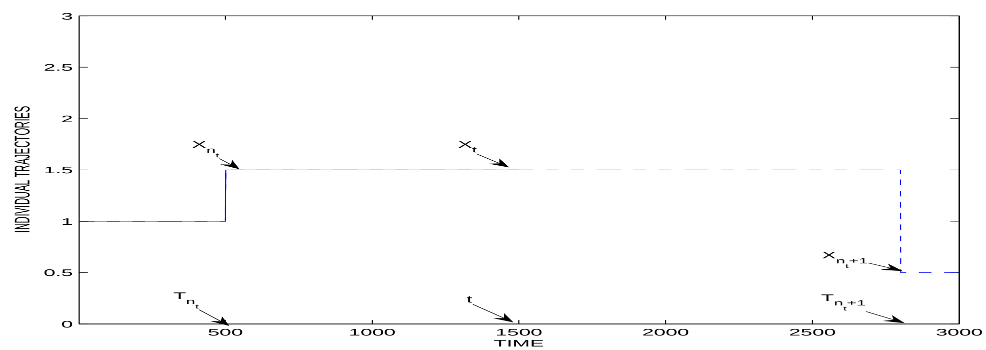
$$\lambda_{i,j}(\tau) \;=\; \lim_{\Delta\tau \to 0} \frac{P(\Delta T_{n+1} \in (\tau, \tau + \Delta\tau), X_{n+1} = j | X_n = i, \Delta T_{n+1} > \tau)}{\Delta\tau}$$

$$\lambda_{i,j}(\tau) \;=\; \frac{\dot{Q}_{i,j}(\tau)}{1 - \sum_j Q_{i,j}(\tau)} \Longleftrightarrow Q_{i,j}(\tau) = \int_0^\tau \lambda_{i,j}(u) exp(-\int_0^u \sum_j \lambda_{i,j}(s) ds) du$$

Particular case : Markov process (no memory) $\Longrightarrow \lambda_{i,j}(\tau) = \lambda_{i,j} = \lambda_i P(i,j)$, for any τ

$F_{i,j}(\tau) = 1 - \exp(-\lambda_i \, \tau) \Longleftrightarrow$

$P(\Delta T_1 > \tau_1 + \tau_2 | \Delta T_1 > \tau_1, X_1 = j, X_0 = i) = P(\Delta T_1 > \tau_2 | \Delta T_1 > 0, X_1 = j, X_0 = i)$



Event-driven simulation algorithm

Current jump time and jump state : $(t_n, i) \Longrightarrow$ determine the next jump time and jump state (t_{n+1}, j)

Simulate j according to $\{P(i, j')\}_j'$

Simulate $t_{n+1} - t_n = \tau$ according to $F_{i,j}(.)$

Use $F(x) \stackrel{def.}{=} P(X < x) = P(U < u)$, $U = F(X)$, $u = F(x)$, U uniform on $(0, 1)$

Probability law of the process (renewal equations)

$$P(X_t = j|X_0 = i) \;=\; P(\Delta T_1 > t|X_0 = i)\delta_{i,j} + \sum_{k\in\mathcal{X}} \int_0^t dP(X_s = k, \Delta T_1 = s|X_0 = i)P(X_t = j|X_s = k)$$

$$P_{i,j}(t) \;=\; [1 - \sum_{j'} Q_{i,j'}(t)]\delta_{i,j} + \sum_k \int_0^t dQ_{i,k}(s)P_{k,j}(t-s)$$

Notations : $\mathbf{P}[i,j] = P_{i,j}(.)$, $\mathbf{Q}[i,j] = Q_{i,j}(.)$, $(\mathbf{I} - \mathbf{Q}^{\Sigma})[i,j] = [1 - \sum_{j'} Q_{i,j'}(t)]\delta_{i,j}$

$$\mathbf{P} \;=\; (\mathbf{I} - \mathbf{Q}^{\Sigma}) + \mathbf{Q} * \mathbf{P} \Longrightarrow \mathbf{P} = \sum_{n=0}^{\infty} \mathbf{Q}^{n*} * (I - \mathbf{Q}^{\Sigma})$$

Approximate solutions

Empirical distribution (simulations)

$\mathbf{P} \simeq \sum_{n=0}^{n_t} \mathbf{Q}^{*n} * (I - \mathbf{Q}^{\Sigma})$

Recursive solution of $P_{i,j}(nh) = [1 - \sum_{j'} Q_{i,j'}(nh)]\delta_{i,j} + \sum_k \sum_{i=1}^{n-1} a_i \dot{Q}_{i,k}(ih)P_{k,j}((n-i)h)$, $t = nh$, $n \geq 1$

Upper and lower bounds of Li and Luo (2005)

Stationary law or quasi-stationary law

Stationary law

$\{X_t\} \Longleftrightarrow \{X_n, T_n\} \Longleftrightarrow Q_{i,j}(.) = F_{i,j}(.)P(i,j)$

Assumptions

1. $\{P(i,j)\}$ is recurrent (all return times are finite) $\Longrightarrow \exists!$ invariant law $\nu = \nu P$

$\Longrightarrow \lim_{n\to\infty} P(X_n = j|X_0 = i) = \nu_j$ (aperiodic case)

2. $0 < m_i < \infty$, where $m_i \stackrel{def.}{=} E(\Delta T_1|X_0 = i)$.

$\Longrightarrow \lim_{t\to\infty} P(X_t = j|X_0 = i) = \nu_j m_j[\sum_k \nu_k m_k]^{-1}$ (aperiodic case)

Quasi-stationary law

Assumptions

1. There exists an absorbing state a and $\{P^c(i,j) \stackrel{def.}{=} P(i,j)/(1 - P(i,a))\}$ is recurrent

$\Longrightarrow \lim_{n\to\infty} P(X_n^c = j|X_0^c = i) = \nu_j^c$

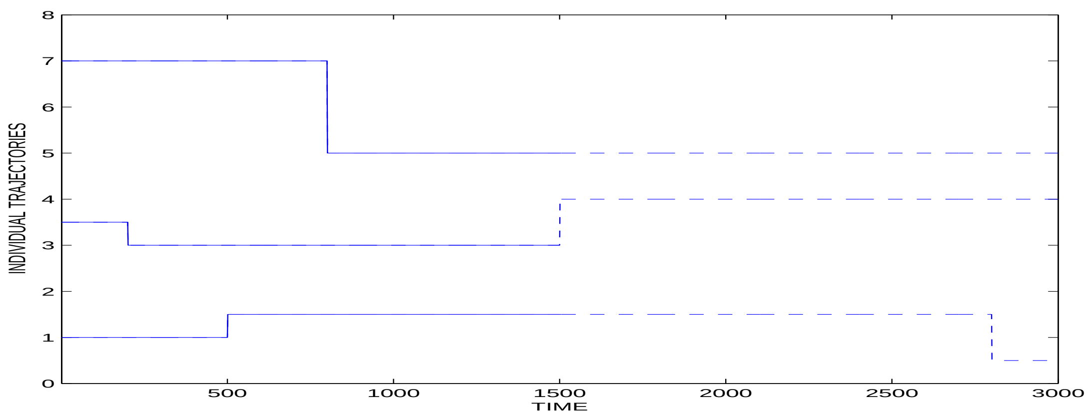
2. $0 < m_i^c < \infty \Longrightarrow \lim_{t\to\infty} P(X_t^c = j|X_0^c = i) = \nu_j^c m_j^c[\sum_k \nu_k^c m_k^c]^{-1}$ (aperiodic case)

Homogeneous Semi-Semi-Markov Processes for a closed population $\Omega = \{\omega_l\}_{l \leq N}$

Set of MRP : $\{\{(X_m^{(l)}(\omega_l), T_m^{(l)}(\omega_l))\}_m\}_{l=1,\ldots,N}$

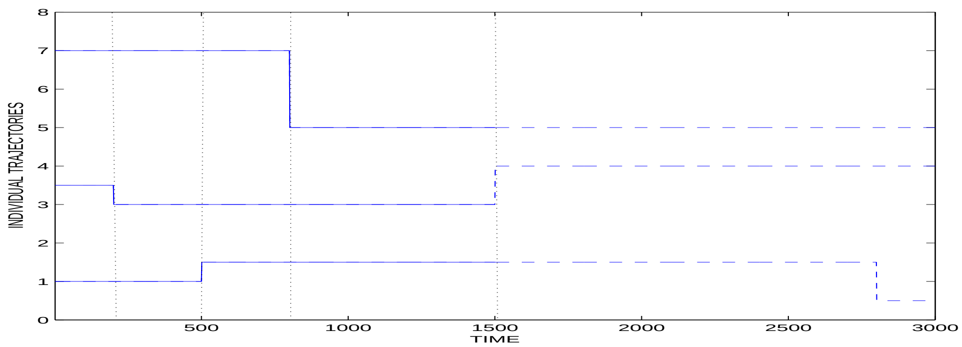
Goal : determine the distribution of the population process $\mathcal{X}_t = \{X_t^{(1)}, \ldots, X_t^{(N)}\}$

Difficulty : the individual MRP are not synchronized and may be population-dependent



Particular case : the MRP are i.i.d. $\Longrightarrow$ asymptotic distributions in the heavy-tailed case (usual case $1 - F(t) \leq O(t^{-1-\alpha})$; heavy-tailed case : $1 - F(t) = t^{-\alpha} L(t)$)
(communication networks (Mikosh and Resnick, 2005, Mitov and Yanev, 2006))

Definition of a SSMP



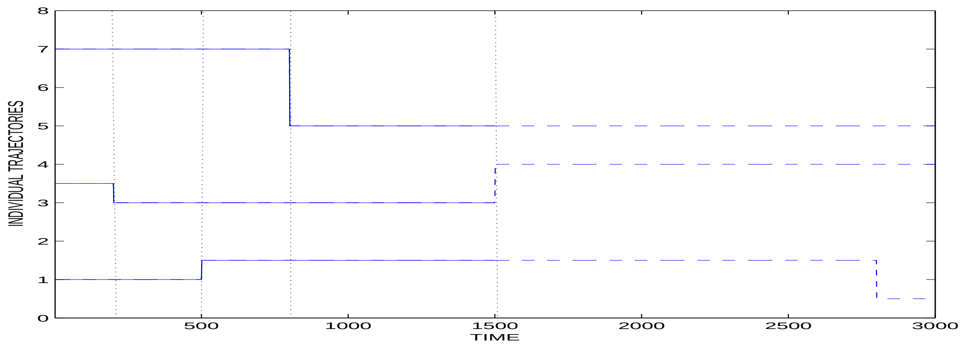
$$\mathcal{X}_t(\Omega) \overset{def.}{=} \mathcal{X}_{n_t}(\Omega) \overset{def.}{=} \{X^{(l)}_{m_{l,t}}(\omega_l)\}_l$$

$$m_{l,t}(\omega_l) \overset{def.}{=} \sup\{m : T^{(l)}_m(\omega_l) \leq t\}$$

$$n_t(\Omega) \overset{def.}{=} \sum_{(l)} m_{l,t}(\omega_l)$$

$$\mathcal{T}_{n_t} \overset{def.}{=} \sup_{(l)} \sup_{m} \{T^{(l)}_m(\omega_l) \leq t\}$$

Consequence : $\{\mathcal{T}_n\}_n = \{T^{(l)}_m\}_{l,m}$



Law of $\{\mathcal{X}_t(\Omega)\}_t \Longleftrightarrow$ law of $\{\mathcal{X}_n, \mathcal{T}_n\}_n \Longleftrightarrow \{P(\mathcal{X}_{n+1} = J, \Delta\mathcal{T}_{n+1} \leq \tau | \mathcal{F}_n(I))\}_{n,I,J}$

Notation : $\mathcal{F}_n(I) = \{\mathcal{X}_n = I, \mathcal{X}_{n-1} = I_{n-1}, ..., \mathcal{X}_0 = I_0, \mathcal{T}_n = t_n, \mathcal{T}_{n-1} = t_{n-1}, ..., \mathcal{T}_0 = t_0\}$

Kernel

$$\begin{aligned} P(\mathcal{X}_{n+1} = J, \Delta\mathcal{T}_{n+1} \leq \tau | \mathcal{F}_n(I)) &= P(\Delta\mathcal{T}_{n+1} \leq \tau | \mathcal{X}_{n+1} = J, \mathcal{F}_n(I)) P(\mathcal{X}_{n+1} = J | \mathcal{F}_n(I)) \\ &\overset{notation}{=} F_{\mathcal{F}_n(I),J}(\tau) P(\mathcal{F}_n(I), J) \\ &\overset{notation}{=} Q_{\mathcal{F}_n(I),J}(\tau) \end{aligned}$$

Goal : calculate $F_{\mathcal{F}_n(I),J}(.)$, $P(\mathcal{F}_n(I), J)$ from the individual transitions

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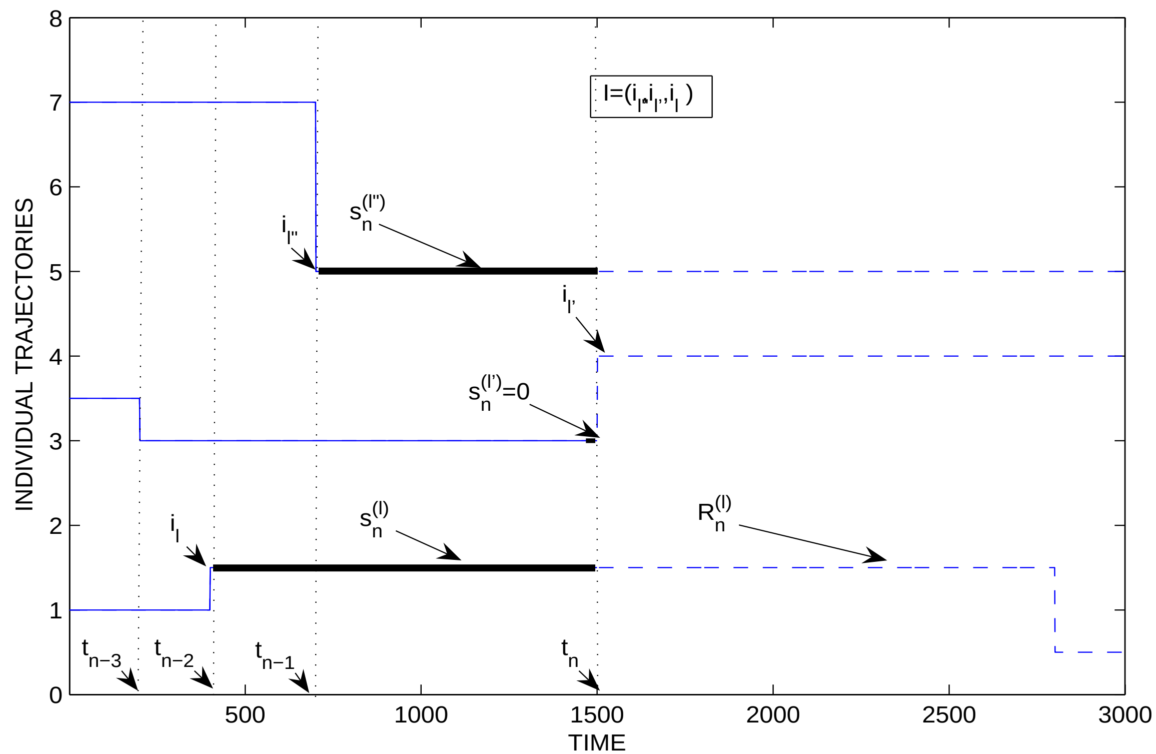
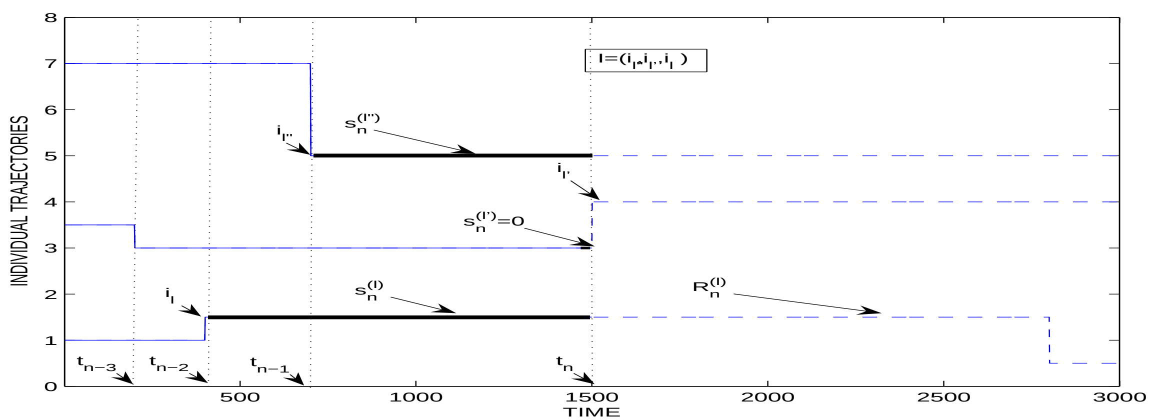


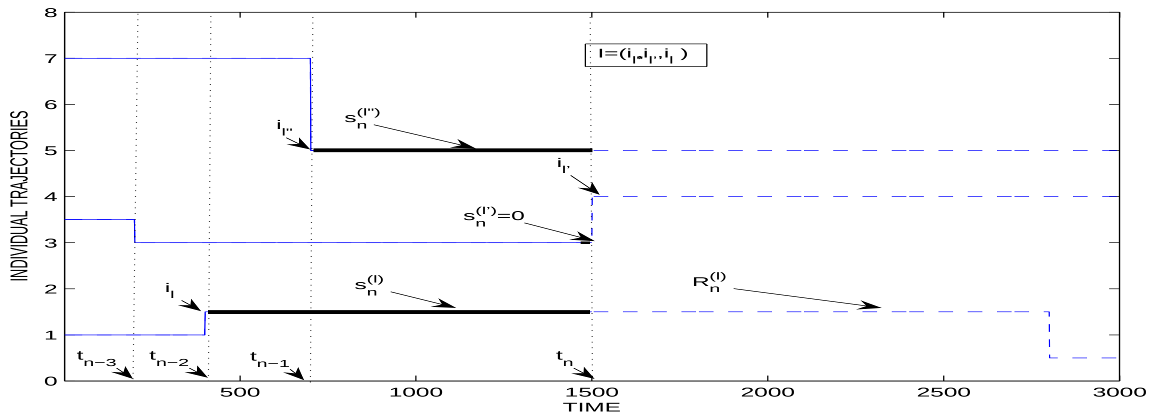
Assumptions given $\mathcal{F}_n(I)$ (past until t_n with $\mathcal{X}_{t_n} = I = (i_1, ..., i_l, ..., i_N)$)

1. $A1$: the $\{(residual\, waiting\, time\; R_n^{(l)},\;\; jump\, state\; X_{m_n+1}^{(l)})\}_l$ are mutually independent
2. $A2$: for each l, $R_n^{(l)}$ is also independent of $\{s_n^{(l')}\}_{l' \neq l}$

$$\Longrightarrow F_{i_l|I,j_l}^{(l)|s_n^{(l)}}(\tau) = \frac{F_{i_l|I,j_l}^{(l)}(s_n^{(l)} + \tau) - F_{i_l|I,j_l}^{(l)}(s_n^{(l)})}{1 - F_{i_l|I,j_l}^{(l)}(s_n^{(l)})}; \quad F_{i_l|I,j_l}^{(l)|0}(\tau) = F_{i_l|I,j_l}^{(l)}(\tau)$$

3. $A3$: the probability for l to jump from i_l to j_l depends only on i_l, j_l, and I : $P^{(l)}(i_l|I, j_l)$





Individual kernel at t_n given $\mathcal{F}_n(I)$: $Q^{(l')|s_n^{(l')}}_{i_{l'}|I,j_{l'}}(\tau) = F^{(l')|s_n^{(l')}}_{i_{l'}|I,j_{l'}}(\tau)P^{(l')}(i_{l'}|I,j_{l'})$, $l' = 1, ..., N$

Proposition. *Let* $I \to J_l$ *:* $i_l \to j_l$*. Then*

$$dF_{\mathcal{F}_n(I),J_l}(\tau) \;=\; \frac{dQ_{\mathcal{F}_n(I),J_l}(\tau)}{P(\mathcal{F}_n(I),J_l)} = \frac{\int_0^\tau \Pi_{l'\neq l}(1-\sum_{j_{l'}\in\mathcal{X}_{l'}(I)} Q^{(l')|s_n^{(l')}}_{i_{l'}|I,j_{l'}}(\tau))dQ^{(l)|s_n^{(l)}}_{i_l|I,j_l}(\tau)}{\int_0^\infty \Pi_{l'\neq l}(1-\sum_{j_{l'}\in\mathcal{X}_{l'}(I)} Q^{(l')|s_n^{(l')}}_{i_{l'}|I,j_{l'}}(\tau))dQ^{(l)|s_n^{(l)}}_{i_l|I,j_l}(\tau)}$$

$$P(\mathcal{F}_n(I),J_l) \;=\; \int_0^\infty dQ_{\mathcal{F}_n(I),J_l}(\tau) = \int_0^\infty \Pi_{l'\neq l}(1-\sum_{j_{l'}\in\mathcal{X}_{l'}(I)} Q^{(l')|s_n^{(l')}}_{i_{l'}|I,j_{l'}}(\tau))dQ^{(l)|s_n^{(l)}}_{i_l|I,j_l}(\tau),$$

Consequence : $Q_{\mathcal{F}_n(I),J_l}(.) = Q_{\{s_n^{(l')}\}_{l'};I,J_l}(.)$

Corollary. Assume Exp *:* $F^{(l)}_{i_l|I,j_l}(\tau) = 1 - \exp(-\lambda_{i_l|I}\,\tau)$
Then the SSMP is a MP, and for all I *not absorbing*

$$
\begin{aligned}
dF_{I,J_l}(\tau) &= dF_I(\tau) = (\sum_{l'} \lambda_{i_{l'}|I}) \exp(-\sum_{l'} \lambda_{i_{l'}|I}\,\tau) d\tau \\
P(I, J_l) &= P^{(l)}(i_l|I, j_l) \frac{\lambda_{i_l|I}}{\sum_{l'} \lambda_{i_{l'}|I}}.
\end{aligned}
$$

Consequence. Under (Exp), if I is not an absorbing state, then

$$
m_I = \Big[\sum_{l'} \lambda_{i_{l'}|I}\Big]^{-1}
$$

Simulation algorithm

Time-driven simulation algorithm $\Longleftrightarrow \lambda_{\mathcal{F}_n(I),J}(.)$

Event-driven simulation algorithm $\Longleftrightarrow Q_{\mathcal{F}_n(I),J}(.) = F_{\mathcal{F}_n(I),J}(.)P(\mathcal{F}_n(I), J)$

Assume $I = (i_1, ..., i_l, ..., i_N)$ at time t_n

Determine the next jump J and the corresponding jump time t_{n+1} :

For each $i_l \in I$ such that the next state j_l and the jump time $t^{(l)}_{m_n+1}$ are not yet simulated or such that their transition laws depend not only on i_l but also on the current state I of the population,

1. Choose j_l according to $\{P^{(l)}(i_l|I, j_l)\}_{j_l}$
2. Simulate according to the law of $R^{(l)}_n$ a residual waiting time $r^{(l)}_n$ in i_l from t_n until the jump into j_l ; deduce $t^{(l)}_{m_n+1} = r^{(l)}_n + t_n$.
3. Keep the simulations $\{j_l, t^{(l)}_{m_n+1}\}_l$ in memory.

Then the minimum time $t^{(l)}_{m_n+1}$ among the set of all simulated jump times $\{t^{(l')}_{m_n+1}\}_{l'}$ defines the next jump time and the next state $J_l = (i_1, ..., j_l, ..., i_N)$.

Transition rates

$$\lambda_{\mathcal{F}_n(I),J}(\tau) \overset{definition}{=} \lim_{\Delta\tau \to 0} \frac{P(\Delta\mathcal{T}_{n+1} \in (\tau, \tau + \Delta\tau), \mathcal{X}_{n+1} = J | \mathcal{F}_n(I), \Delta\mathcal{T}_{n+1} > \tau)}{\Delta\tau}$$

Proposition

1. *Assume a continuous time with* $\{dF^{(l)}_{i_l|I,j_l}(\tau)/d\tau\}$. *Then, for* $\tau \in \mathbb{R}^+$,

$$\begin{aligned} \lambda_{\mathcal{F}_n(I),J}(\tau) &= \frac{\dot{Q}_{\mathcal{F}_n(I),J}(\tau)}{1 - \sum_J Q_{\mathcal{F}_n(I),J}(\tau)} \\ Q_{\mathcal{F}_n(I),J}(\tau) &= \int_0^\tau \lambda_{\mathcal{F}_n(I),J}(u) exp(- \int_0^u \sum_J \lambda_{\mathcal{F}_n(I),J}(s) ds) du \end{aligned}$$

2. *Assume a discrete time with a time step* $\Delta\tau = 1$. *Then, for* $\tau \in \mathbb{N}$,

$$\begin{aligned} \lambda_{\mathcal{F}_n(I),J}(\tau) &= \frac{Q_{\mathcal{F}_n(I),J}(\tau + 1) - Q_{\mathcal{F}_n(I),J}(\tau)}{1 - \sum_J Q_{\mathcal{F}_n(I),J}(\tau)} \\ Q_{\mathcal{F}_n(I),J}(\tau + 1) &= \sum_{l=0}^{\tau} \lambda_{\mathcal{F}_n(I),J}(l) \Pi_{k=0}^{l-1} [1 - \sum_{J'} \lambda_{\mathcal{F}_n(I),J'}(k)]. \end{aligned}$$

Consequence : $\lambda_{\mathcal{F}_n(I),J}(\tau) \overset{not.}{=} \lambda_{\{s_n^{(l')}\}_{l'};I,J}(\tau)$

Particular case (Exp) : $F^{(l)}_{i_l|I,j_l}(\tau) = 1 - \exp(-\lambda_{i_l|I}\,\tau)$

Corollary. *The population transition rate is (MP case)*

$$\begin{aligned} \lambda_{I,J_l}(\tau) &= \lambda_{i_l|I} P^{(l)}(i_l|I,j_l) = \lambda_{i_l|I,j_l} \\ \lambda_I(\tau) &= \sum_J \lambda_{I,J}(\tau) = \sum_l \lambda_{i_l|I} \end{aligned}$$

Marginal probability law of $\{\mathcal{X}_t\}_t$: renewal equations

Proposition. *Let* $D_{n,t} = \{t_1, ..., t_n : 0 < t_1 < ... < t_n \leq t\}$*. Then*

$$P(\mathcal{X}_t = J|\mathcal{F}_0(I_0)) = P(\sqcup_n\{\mathcal{X}_{\mathcal{T}_n} = J, \mathcal{T}_n \leq t, \mathcal{T}_{n+1} > t\}|\mathcal{F}_0(I_0)) =$$

$$\sum_n \sum_{\{I_h\}_{h\leq n}} \int_{D_{n,t}} dP(\{(\mathcal{X}_{\mathcal{T}_h} = I_h, \mathcal{T}_h = t_h)\}_{h=1,n}, \mathcal{T}_{n+1} > t|\mathcal{F}_0(I_0))\delta_{J,I_n} =$$

$$\sum_n \sum_{\{I_h\}_{h\leq n}} \int_{D_{n,t}} P(\mathcal{T}_{n+1} > t|\mathcal{F}_n(I_n))\Pi_{h=1}^n dP(\mathcal{X}_h = I_h, \mathcal{T}_h = t_h|\mathcal{F}_{h-1}(I_{h-1}))\delta_{J,I_n} =$$

$$\sum_n \int_{t_0<t_1\leq t} \sum_{I_1} dQ_{\{s_0^{(l)}\}_l;I_0,I_1}(t_1 - t_0) ... \int_{t_{n-1}<t_n\leq t} \sum_{I_n} dQ_{\{s_{n-1}^{(l)}\}_l;I_{n-1},I_n}(t_n - t_{n-1})[1 - Q_{\{s_n^{(l)}\}_l;I_n}(t - t_n)]\delta_{J,I_n}$$

where $s_h^{(l)} = [s_{h-1}^{(l)} + t_h - t_{h-1}]1_{\{t_h \notin \{T_m^{(l)}\}_m\}}$

Notation : $\mathbf{Q}_h[I, J](t) = Q_{\{s_h^{(l)}\}_l;I,J}(T)$

$$\mathbf{P}_0 = (\mathbf{I} - \mathbf{Q}_0^\Sigma) + \sum_{n\geq 1} \mathbf{Q}_0 \circ \mathbf{Q}_1 \circ ... \circ \mathbf{Q}_{n-1} \circ (\mathbf{I} - \mathbf{Q}_n^\Sigma) \quad (1)$$

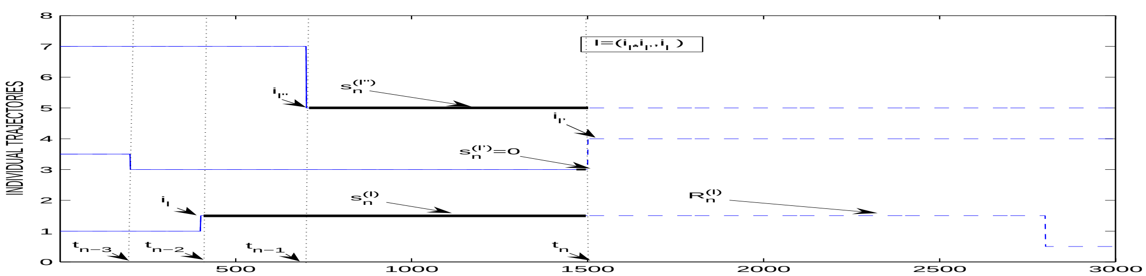
$$P(\mathcal{X}_t = J|\mathcal{F}_0(I_0)) =$$

$$\sum_n \int_{t_0<t_1\leq t} \sum_{I_1} dQ_{\{s_0^{(l)}\}_l;I_0,I_1}(t_1-t_0)\ldots \int_{t_{n-1}<t_n\leq t} \sum_{I_n} dQ_{\{s_{n-1}^{(l)}\}_l;I_{n-1},I_n}(t_n-t_{n-1})[1-Q_{\{s_n^{(l)}\}_l;I_n}(t-t_n)]\delta_{J,I_n}.$$

Renewal equations

$$s_h^{(l)} = [s_{h-1}^{(l)} + t_h - t_{h-1}]1_{\{t_h\notin\{T_m^{(l)}\}_m\}},\ \{s_h^{(l)}\}_l = \Delta_{t_h-t_{h-1}};\ \{s_0^{(l)}\}_l = \Delta_0$$

$$\mathbf{P}_{\Delta_0}(t-t_0) = \mathbf{I} - \mathbf{Q}^{\Sigma}_{\Delta_0}(t-t_0) + \int_{t_1\in(t_0,t)} d\mathbf{Q}_{\Delta_0}(t_1-t_0)\mathbf{P}_{\Delta_{t_1-t_0}}(t-t_1). \quad (2)$$



Approximate solution of the renewal equation

$$\mathbf{P}_{\Delta_0}(t-t_0) = \mathbf{I} - \mathbf{Q}^{\Sigma}_{\Delta_0}(t-t_0) + \int\limits_{t_1\in(t_0,t)} d\mathbf{Q}_{\Delta_0}(t_1-t_0)\mathbf{P}_{\Delta_{t_1-t_0}}(t-t_1).$$

Corollary The discretization of the renewal system using $t-t_0=nh$*,* $t_1-t_0\in\{ih\}_{i\leq n}$*, leads to the solution*

$$\begin{pmatrix} \mathbf{P}_{\Delta_0}(nh) \\ \\ \mathbf{P}_{\Delta_h}((n-1)h) \\ \\ ... \\ \\ \mathbf{P}_{\Delta_{(n-1)h}}(h) \end{pmatrix} = \begin{pmatrix} \mathbf{R}_{\Delta_0}(0) & \mathbf{R}_{\Delta_0}(h)\ldots\mathbf{R}_{\Delta_0}((n-1)h) \\ 0 & \mathbf{R}_{\Delta_h}(0)\ldots\mathbf{R}_{\Delta_h}((n-2)h) \\ . & .\ \ \ldots\ldots\ldots \\ 0 & 0\ldots\ldots\ldots\mathbf{R}_{\Delta_{(n-1)h}}(0) \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{B}_{\Delta_0,n} \\ \\ \mathbf{B}_{\Delta_h,n} \\ \\ ... \\ \\ \mathbf{B}_{\Delta_{(n-1)h},n} \end{pmatrix}$$

$\mathbf{R}_{\Delta}(ih) = \mathbf{I}\delta_{0,i} - a_i\dot{\mathbf{Q}}_{0,\Delta}(ih)(1-\delta_{0,i})$, $\delta_{0,i}=1$ *when* $i=0$ *(and is 0 otherwise)*, $i=0,\ldots,n-1$
$\mathbf{B}_{\Delta_{jh},n} = \mathbf{I} - \mathbf{Q}^{\Sigma}_{\Delta_{jh}}((n-j)h)$, $j=0,\ldots,n-1$
$\{a_i\}_i$ *depends on the numerical integration scheme,*

Proof

Let $t - t_0 = nh$, $t_1 - t_0 \in \{ih\}_{i \leq n}$.

$$\mathbf{P}_{\Delta_0}(nh) = \mathbf{I} - \mathbf{Q}^{\Sigma}_{\Delta_0}(nh) + \sum_{i=1}^{n-1} a_i \dot{\mathbf{Q}}_{\Delta_0}(ih)\mathbf{P}_{\Delta_{ih}}((n-i)h)$$

equivalent to

$$\begin{aligned} \mathbf{P}_{\Delta_0}(nh) - \sum_{i=1}^{n-1} a_i \dot{\mathbf{Q}}_{\Delta_0}(ih)\mathbf{P}_{\Delta_{ih}}((n-i)h) &= \mathbf{I} - \mathbf{Q}^{\Sigma}_{\Delta_0}(nh) \\ \sum_{i=0}^{n-1} [\mathbf{I}\delta_{0,i} - a_i \dot{\mathbf{Q}}_{\Delta_0}(ih)(1-\delta_{0,i})]\mathbf{P}_{\Delta_{ih}}((n-i)h) &= \mathbf{I} - \mathbf{Q}^{\Sigma}_{\Delta_0}(nh), \end{aligned} \quad (3)$$

Use $\mathbf{R}_{\Delta_0}(ih) = \mathbf{I}\delta_{0,i} - a_i \dot{\mathbf{Q}}_{\Delta_0}(ih)(1-\delta_{0,i})$ and (3) with $(n-j, \Delta_{jh})$ instead of (n, Δ_0), $j = 0, \ldots, n-1$

$$\Longrightarrow \sum_{i=0}^{n-j-1} \mathbf{R}_{\Delta_{jh}}(ih)\mathbf{P}_{\Delta_{(j+i)h}}((n-j-i)h) = \mathbf{I} - \mathbf{Q}^{\Sigma}_{\Delta_{jh}}((n-j)h)$$

Particular case : propagation of a disease on a graph with N vertices (farms or plants)

N=2 $\Longrightarrow$ population states : $(0,0)$ (absorbing), $(0,1)$, $(1,0)$, $(1,1)$

Individual kernels

$P^{(1)}(0|I,1) = 1$ if $I \supset 1$ (probability of activation)

$P^{(1)}(0|I,1) = 0$ if $I = (0,0)$

$P^{(1)}(1|I,0) = 1$ (probability of inactivation)

$F^{(l)}_{i_l|I,j_l}(\tau) = 1 - \exp(-\lambda_{i_l|I}\tau)$ (cdf of the time of the transition $i_l|I \rightarrow j_l$)

$\lambda_{0|I} \stackrel{ex.}{=} \lambda_{0||I|}$ (activation rate)

$\lambda_{1|I} \stackrel{ex.}{=} \lambda_{1||I|}$ (inactivation rate)

Population process

$dF_{I,J_l}(\tau) = (\sum_l \lambda_{i_l|I}) \exp(-\sum_l \lambda_{i_l|I}\, \tau) d\tau$

$P(I, J_l) = P^{(l)}(i_l|I, j_l) \frac{\lambda_{i_l|I}}{\sum_{l'} \lambda_{i_{l'}|I}}$

Censored population process

A unique absorbing state $0 = (0, 0)$

$$dF_{I,J_l}(\tau) = (\sum_l \lambda_{i_l|I}) \exp(-\sum_l \lambda_{i_l|I}\, \tau) d\tau$$

$$\begin{aligned} P^c(I, J_l) &= \frac{P(I, J_l)}{1 - P(I, 0)}, \quad P(I, J_l) = P^{(l)}(i_l|I, j_l)\frac{\lambda_{i_l|I}}{\sum_{l'} \lambda_{i_{l'}|I}} \\ Q^c_{I,J_l}(\tau) &= F_{I,J_l}(\tau)P^c(I, J_l), \quad Q^c_I(\tau) = \sum_J Q^c_{I,J}(\tau) \end{aligned}$$

$\Longrightarrow \{P^c(I, J) = P(I, J)/(1 - P(I, 0))\}$ is recurrent $\Longrightarrow \lim_{n\to\infty} P(\mathcal{X}_n = J|\mathcal{X}_0 = I) = \nu^c_J$, where $\nu^c P^c = \nu^c$

For $I \neq (0, 0)$, $m_I = \left[\sum_{l'} \lambda_{i_{l'}|I}\right]^{-1} \Longrightarrow 0 < m_I < \infty$

$\Longrightarrow \lim_{t\to\infty} P(\mathcal{X}_t = J|\mathcal{X}_0 = I) = \nu^c_J m^c_J [\sum_K \nu^c_K m_K]^{-1}$ (aperiodic case)

Conclusion

– Formalization of individual-based models at the population level $\Longrightarrow$ SSMP
$\Longrightarrow$ kernel $\Longrightarrow$ law of the process (renewal equations), simulation algorithm, asymptotic law ?

– Generalization of individual-based models to branching populations

– Particular case (Exp) : $F^{(l)}_{i_l|I,j_l}(\tau) = 1 - \exp(-\lambda_{i_l|I}\,\tau)$, the SSMP is a MP $\Longrightarrow$ more results

THANK YOU FOR YOUR ATTENTION !

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