Continuum Percolation in the Relative Neighborhood Graph

Jean-Michel Billiot, Franck Corset and Eric Fontenas

Laboratoire Jean Kuntzmann, Département de Statistique, équipe FIGAL
Université de Grenoble

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Outline

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Goal of this talk

Existence of nontrivial site percolation threshold in the Relative Neighborhood Graph (RNG) for Poisson stationary point process with unit intensity in $\mathbb{R}^2$.

Some applications

- Ferromagnetism (at low temperature) and Ising model
- Disordered electrical networks (electrical resistance of a mixture of two materials)
- Cancerology for the study of the growth of tumor when the cancer cells suddently begin to invade healthy tissue.
- Epidemics and fires in orchards

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Some graphs

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The Delaunay graph $Del_2(\varphi)$ of a configuration φ in Ω is the set of edges of the unique triangulation $Del_3(\varphi)$ in which the interior of the circle circumscribed by every triangle of $Del_3(\varphi)$ does not contain any point of φ.

Definition

The Gabriel graph $Gab(\varphi)$ of a configuration φ in Ω is defined as the set of edges $\{u, v\} \subset \varphi$ such that the open circle with $\{u, v\}$ as diameter does not contain any point of the configuration φ.

These graphs are planar in $\mathbb{R}^2$.

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Relative Neighborhood Graph

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The Relative Neighborhood graph $Rn(\varphi)$ of a configuration φ in Ω is defined as the set of edges $\{u, v\} \subset \varphi$ such that the intersection of the disks with center u and v with radius uv does not contain any point of the configuration φ.

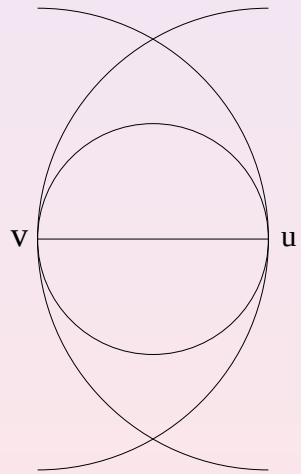

Figure: Vacuity regions for Gabriel and RN graphs for an edge uv

The Model

- Site percolation on the RNG for the Poisson point process Φ
- Bernoulli process $\Theta^p(\Phi)$ providing the type picking mechanism (1 for open with probability p and 0 for closed with probability $1-p$) of the points (or sites) in Φ.
- $\Phi^p = (\Phi, \Theta^p(\Phi))$ can be seen as a marked Poisson process
- The probability measure of Φ^p is given by :

$$\mathbb{P}(d\varphi, d\xi) = \int \Pi(d\varphi)\mu_\varphi^p(d\xi)$$

where μ_φ^p is the probability measure on $\{0,1\}^\varphi$ of the Bernoulli process $\Theta^p(\varphi)$ given the configuration $\varphi \in \Omega$.

- $\mu_{Rn(\varphi)}^p$ is the probability measure on $\{0,1\}^{Rn(\varphi)}$ of the Bernoulli process $\Theta^p(Rn(\varphi))$ given the graph $Rn(\varphi)$ where $\varphi \in \Omega$.

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Critical Probabilities

We define $p_c^{site}(\varphi, Rn(\varphi))$ and $p_c^{bond}(\varphi, Rn(\varphi))$ such that:

$$\mu^p_{Rn(\varphi)}(\exists \text{ at least one infinite open cluster in } Rn(\varphi))$$

$$= \begin{cases} 1 & \text{if} \quad p > p_c^{site}(\varphi, Rn(\varphi)) \\ 0 & \text{if} \quad p < p_c^{site}(\varphi, Rn(\varphi)) \end{cases}$$

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Proposition

$\forall \varphi \in \Omega$, as $(\varphi, Rn(\varphi))$ has a maximum vertex degree equals to 6,

$$\frac{1}{6-1} \leq p_c^{bond}(\varphi, Rn(\varphi)) \leq p_c^{site}(\varphi, Rn(\varphi)) \leq 1 - [1 - p_c^{bond}(\varphi, Rn(\varphi))]^6.$$

Theorem

$p_c^{\text{site}}(Rn, \Pi) < 1$ *and* $p_c^{\text{bond}}(Rn, \Pi) < 1$.

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Sktech of proof

To prove that continuous percolation occurs, we shall compare the process to various bond percolation models on Z^2. In these models, the states of the edges are not be independent.

De nition

A bond percolation model is 1-independent if whenever E_1 and E_2 are sets of edges at graph distance at least 1 from each another (i.e., if no edge of E_1 is incident to an edge of E_2) then the state of the edges in E_1 is independent of the state of the edges in E_2.

We proceed in two steps:

1. Control the probability of some suitable con gurations of points under Poisson point process.
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The Rolling Ball Method

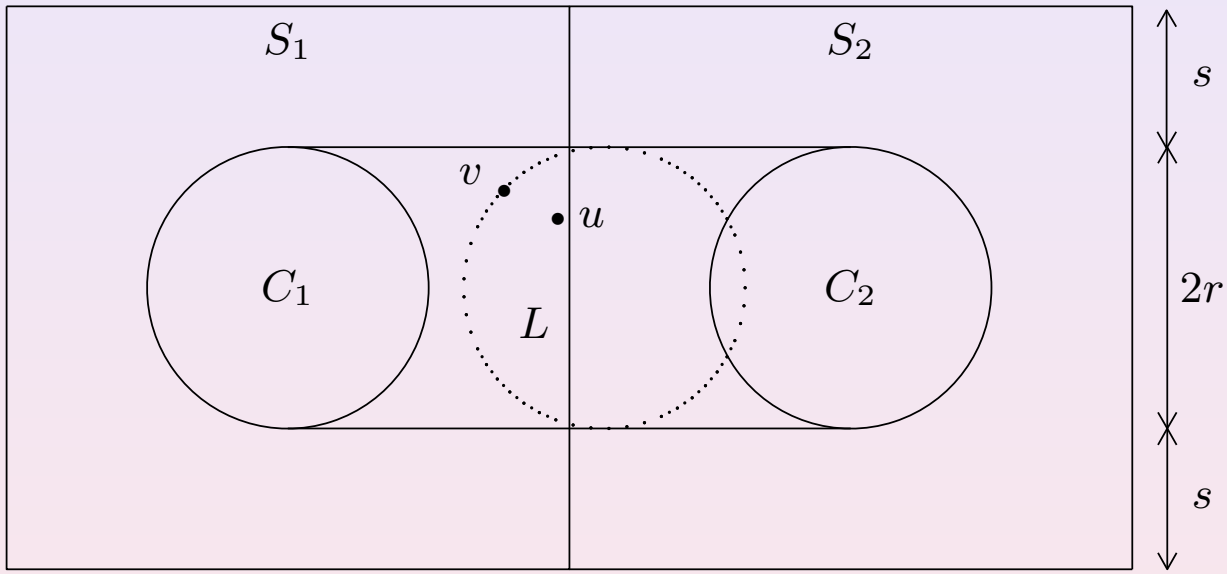

Figure 1: The Rolling Ball Method

Comparison with $\mathbb{Z}^2$

- Write $u \sim v$ if uv is an edge of the underlying graph $Rn(\varphi)$
- Percolation = infinite path: a sequence $u_1, u_2 \ldots$ with $u_i \sim u_{i+1}$ for all i.
- Let $\mathcal{E}_{S_1,S_2}$ be the event that every vertex u_1 in the central disk C_1 of S_1 is joined to at least one vertex v in the central disk C_2 of S_2 by a $Rn-$ path, regardless of the state of the Poisson process outside of S_1 and S_2.
- Each vertex $(i, j) \in \mathbb{Z}^2$ corresponds to a square $[Ri, R(i+1)] \times [Rj, R(j+1)] \in \mathbb{R}^2$, where $R = 2r + 2s$, and an edge is open between adjacent vertices (corresponding to squares S_1 and S_2) if both events $\mathcal{E}_{S_1,S_2}$ and $\mathcal{E}_{S_2,S_1}$ hold.
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- Any open path in $\mathbb{Z}^2$ corresponds to a sequence of events $\mathcal{E}_{S_1,S_2}, \mathcal{E}_{S_2,S_3} \ldots$ that occur, where S_i is the square associated with a site in $\mathbb{Z}^2$.
- Every vertex u_1 of the original Poisson process that lies in the central disk C_1 of S_1 now has an infinite path leading away from it, since one can find points u_i in the central disk of S_i and paths from u_{i-1} to u_i inductively for every $i > 1$.
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A result of a 1-independent bond percolation on $\mathbb{Z}^2$

Theorem (Ballister, Bollobas, Walters. *Random Structures and Algorithms*, 2005)

If every edge in a 1-independent bond percolation model on $\mathbb{Z}^2$ is open with probability at least 0.8639*, then almost surely there is an infinite open component. Moreover, for any bounded region, there is almost surely a cycle of open edges surrounding this region.*

The Rolling Ball Method

Let $\mathbf{E}_{S_1,S_2}$ be the event that for every point $v \in C_1 \cup L$, there is a u such that:

a) $v \sim u$;

b) $d(u,v) \leq s$; and

c) $u \in D_v$, where D_v is the disk of radius r inside $C_1 \cup L \cup C_2$ with v on its C_1-side boundary (the dotted disk in Figure 1).

If $\mathbf{E}_{S_1,S_2}$ holds, then every vertex v in C_1 must be joined by a $Rn-$path to a vertex in C_2, since each vertex in $C_1 \cup L$ is joined to a vertex whose disk D_v is further along in $C_1 \cup L \cup C_2$.

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The probability of $\bar{\mathbf{E}}_{S_1,S_2}$ is bounded by the expected number of points u for which above conditions (a)-(c) fail. Thus, we have

$$\Pi(\bar{\mathbf{E}}_{S_1,S_2}) \leq 2r(2r+2s)p_{Rn,r,s} \tag{1}$$

where $p_{Rn,r,s}$ is the probability that (a)-(c) fail for some fixed v. Notice that this probability is independent of the location of v in $C_1 \cup L$.

Lemma

We can choose r and s such that $\Pi(\bar{\mathbf{E}}_{S_1,S_2})$ is arbitrary small.

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sketch of proof

Let $D(x, \alpha)$ be the disk of radius α and of center x. Then,

$$p_{Rn,\,r,\,s} \leq e^{-|D_v \cap D(v,s)|} + \int_{D_v \cap D(v,s)} p_{r,\,s}(u)\, du$$

where $p_{r,\,s}(u)$ is the probability that u is the closest point to v inside D_v, but that (u, v) is not an edge of the RNG.
The first term being the probability that there is no u satisfying (b) and (c), and the integral gives the probability that such a u exists, but that the closest one to v fails (a).

Second Step

We also need to control the probability of having at least one point in C_1 and less than m points in $C_1 \cup L \cup C_2$.

We denote $F_{C_1} = \{\Phi(C_1) \geq 1\}$ and $A_m = \{\Phi(C_2 \cup C_1 \cup L) \leq m\}$. We obtain that

$$\Pi(\mathcal{E}_{s_1,s_2} \cap \mathcal{E}_{s_2,s_1}) \geq 1 - \left[\Pi(\bar{\mathbf{E}}_{s_1,s_2}) + \Pi(\bar{\mathbf{E}}_{s_2,s_1}) + \Pi(\bar{F}_{C_1}) + \Pi(\bar{F}_{C_2}) + \Pi(\bar{A}_m)\right]$$

where $\Pi(\bar{F}_{C_1}) = \Pi(\bar{F}_{C_2}) = e^{-\pi r^2}$ and

$$\Pi(\bar{A}_m) = \sum_{k>m} \frac{[2r(2r+2s) + \pi r^2]^k}{k!} e^{-2r(2r+2s)-\pi r^2}.$$

Choosing $r = s$, we have the following bound

$$\Pi(\bar{A}_m) \leq \frac{[(8+\pi)r^2]^{m+1}}{(m+1)!}.$$

Using Stirling's formula, this bound becomes negligible whenever $r \gg 1$ and $m > e(8+\pi)r^2$.

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let $B_{r,s}$ be the event that all the sites are open in $C_2 \cup C_1 \cup L$ with probability $p = 1 - \frac{\epsilon}{2m}$ and also define $C_{r,s} = \mathcal{E}_{S_1,S_2} \cap \mathcal{E}_{S_2,S_1} \cap B_{r,s}$.

$$\begin{aligned}
\mathbb{P}(C_{r,s}) &= \int_{\mathcal{E}_{S_1,S_2} \cap \mathcal{E}_{S_2,S_1}} \Pi(d\varphi)\, \mu_\varphi^p(B_{r,s}) \\
&\geq \int_{\mathbf{E}_{S_1,S_2} \cap \mathbf{E}_{S_2,S_1} \cap F_{C_1} \cap F_{C_2} \cap A_m} \Pi(d\varphi)\, \mu_\varphi^p(B_{r,s}) \\
&\geq (1 - \epsilon/2)p^m > 1 - \epsilon = 0.8639
\end{aligned}$$

because we can choose r, s, m (as preceding) such that

$$\Pi(\mathbf{E}_{S_1,S_2} \cap \mathbf{E}_{S_2,S_1} \cap F_{C_1} \cap F_{C_2} \cap A_m) \geq 1 - \epsilon/2$$

We conclude with theorem of Balister et al. that $p_c^{\text{site}}(Rn, \Pi) \leq 1 - \frac{\epsilon}{2m} < 1$.

let $B_{r,s}$ be the event that all the sites are open in $C_2 \cup C_1 \cup L$ with probability $p = 1 - \frac{\epsilon}{2m}$ and also define $C_{r,s} = \mathcal{E}_{S_1,S_2} \cap \mathcal{E}_{S_2,S_1} \cap B_{r,s}$.

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