Unicity of q-Potts measure on a family of self-dual graphs

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The q-Potts Model

Let be G a connected graph with n vertices.
Partition function $Z(G,q,v) = \sum_{\{\sigma_n\}} e^{-\beta H}$;
The Hamiltonian : $H = -J \sum_{<i,j>} \delta_{\sigma_i \sigma_j}$ with
- $\sigma_i = 1 \ldots q$ spin variables on each vertex i.
- $\beta = (k_B T)^{-1}$
- $< i,j >$ denotes pairs of adjacent vertices.

We denote $\sqrt{q}v = e^{\beta J} - 1 = e^{J/(k_B T)} - 1$.
If $v \geq 0 \Leftrightarrow 0 \leq T, J \geq 0$, for the Potts ferromagnet.
If $-1 \leq \sqrt{q}v \leq 0 \Leftrightarrow 0 \leq T, J \leq 0$, for the Potts antiferromagnet.
We call the half plane $Re(v) \geq 0$: the ferromagnetic region.

Pressure and Tutte polynomial

- Pressure function :

$$p(G,q,v) = \lim_{n\to\infty} \ln(Z(G,q,v)^{1/n}).$$

- Relation between partition function and Tutte polynomial

$$Z(G,q,v) = (x-1)(y-1)^n T(G,x,y)$$

on the curve $(x-1)(y-1) = q \Leftrightarrow \begin{cases} x = 1+\sqrt{q}/v \\ y = 1+\sqrt{q}v \end{cases}$

$z = x+y-2 = \sqrt{q}(v+1/v)$.

Particular cases of the Tutte polynomial

The chromatic polynomial

$$P(G,q) = q(-1)^{n+1} T(G, x = 1-q, y = 0)$$

$T(G,1,1)$ number of spanning trees
$T(G,2,1)$ number of spanning forests, (or independent sets)
$T(G,2,2)$ number of spanning subgraphs
$T(G,1,2)$ number of spanning connected subgraphs
$T(G,2,0)$ number of acyclic orientations
$T(G,0,2)$ number of totally cyclic orientations

If $G = G^\star$ then $T(G,x,y) = T(G,y,x)$

Conjecture of Chen, Hu and Wu (1996)

Conjecture

For finite planar self-dual lattices and for square lattice with free or periodic boundary conditions in the thermodynamic limit, the Potts partition zeros in the $Re(v) > 0$ half plane are located on the unit circle $|v| = 1$.

Graph C : a cycle with an edge of multiplicity three. The Tutte polynomial of C on the hyperbola with $q = 16$:

$$T(C, z) = z^3 + 6z^2 + 12z + 218.$$

One not positive real root $(-2 - (210)^{1/3})$ and two conjugated roots with not negative real part $((210)^{1/3}/2 - 2 \pm i(210)^{1/3}\sqrt{3}/2)$.

Useful tools

Let D be a connected open set in $\mathbb{C}$, and let $\alpha_1, \ldots, \alpha_M, \lambda_1, \ldots, \lambda_M$ be analytic functions on D, none of which is identically zero. For each integer $n \geq 0$, let define

$$f_n(z) = \sum_{k=1}^{M} \alpha_k(z)[\lambda_k(z)]^n.$$

$$Z(f_n) = \{z \in D : f_n(z) = 0\}$$

- liminf $Z(f_n) = \{z \in D$: every neighborhood $U \ni z$ has a nonempty intersection with all but finitely many of the sets $Z(f_n)\}$.
- limsup $Z(f_n) = \{z \in D$: every neighborhood $U \ni z$ has a nonempty intersection with infinitely many of the sets $Z(f_n)\}$.

Let k be a dominant subscript z if $|\lambda_k(z)| \geq |\lambda_l(z)|$ for all $l \in \{1 \ldots M\}$.

Beraha Kahane Weiss theorem

Theorem

Let D be a domain in $\mathbb{C}$, and let $\lambda_1, \ldots \lambda_M, \alpha_1, \ldots, \alpha_M$ be analytic functions on D, none of which is identically zero. There do not exist subscripts $k \neq k^{'}$ such that $\lambda_k = \omega \lambda_{k'}$ for some constant ω with $|\omega| = 1$ and such that $\{z \in D : k \text{ is dominant at } z\}$ $(= \{z \in D : k' \text{ is dominant at } z\})$ has nonempty interior. $\forall n \geq 0$,

$$f_n(z) = \sum_{k=1}^{M} \alpha_k(z) \lambda_k(z)^n$$

Then $\liminf Z(f_n) = \limsup Z(f_n)$, z lies in this set if and only if
(a) There is a unique dominant subscript k at z, and $\alpha_k(z) = 0$; or
(b) There are two or more dominant subscripts at z.

Vitali's convergence theorem

Theorem

Let $p_n(z)$ be a sequence of functions, each regular in a region D. Assume that it exists a constant B as $|p_n(z)| \leq B$ for every n and for all $z \in D$. If $p_n(z)$ tends to a limit as $n \longrightarrow \infty$ at a set of points having a limit point inside D, then $p_n(z)$ tends uniformly to a limit in any region bounded by a contour interior to D: the limit therefore being an analytic function of z.

General results

$$f_n(z) = \sum_{k=1}^{M} \Big[\alpha_{a_k}(z)[\lambda_{a_k}^{+}(z)]^n + \beta_{a_k}(z)[\lambda_{a_k}^{-}(z)]^n \Big]$$

where $\lambda_{a_k}^{+}(z)$ and $\lambda_{a_k}^{-}(z)$ are the solutions of

$$X^2 - (z + 2 + a_k)X + z + q + 1 = 0$$

with $a_k \in [0, q]$.

$$\lambda_{a_k}^{\pm}(z) = \frac{1}{2}\left(z + 2 + a_k \pm \sqrt{(z + a_k)^2 - 4(q - a_k)} \right).$$

$\lambda_{a_k}(z)$ be the solution between $\lambda_{a_k}^{+}(z)$ and $\lambda_{a_k}^{-}(z)$ with the greatest magnitude. $\{\alpha_{a_k}, \beta_{a_k},\ k = 1..M\}$ are such that f_n stays a polynomial function in the variable z.

$a_u = \sup_{k=1\ldots M} a_k$ and $a_l = \inf_{k=1\ldots M} a_k$.

$$D_{]-\infty,-a_u-2[} = \{z = c + id; (c,d) \in \mathbb{R}^2,\ c < -a_u - 2\},$$

$$D_{]-a_l,+\infty[} = \{z = c + id; (c,d) \in \mathbb{R}^2,\ c > -a_l\}.$$

Theorem

There exists only one dominant eigenvalue at z:
- $\forall z \in D_{]-a_l,+\infty[} \setminus \{c \in [-a_l, \sup(-a_l, -a_u + 2\sqrt{q - a_u})],\ d = 0\}$, $\lambda_{a_u}(z)$ is the dominant eigenvalue.
- $\forall z \in D_{]-\infty,-a_u-2[} \setminus \{c \in [\inf(-a_u - 2, -a_l - 2\sqrt{q - a_l}), -a_u - 2],\ d = 0\}$, $\lambda_{a_l}(z)$ is the dominant eigenvalue.

Taking $p_n(z) = \dfrac{\ln(f_n(z))}{n}$,

Analyticity of the pressure

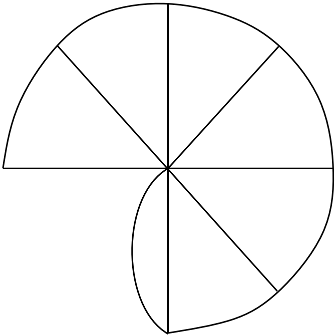
Corollary

We have

- *If* $\forall z \in D_{]-a_l,+\infty[} \setminus \{c \in [-a_l, \sup(-a_l, -a_u + 2\sqrt{q - a_u})], \, d = 0\}$, $\alpha_{a_u}(z) \neq 0$ *then* $p_n(z) \longrightarrow \ln[\lambda_{a_u}(z)]$ *as* $n \to \infty$.
- *If* $\forall z \in D_{]-\infty,-a_u-2[} \setminus \{c \in [\inf(-a_u - 2, -a_l - 2\sqrt{q - a_l}), -a_u - 2], \, d = 0\}$, $\beta_{a_l}(z) \neq 0$ *then* $p_n(z) \longrightarrow \ln[\lambda_{a_l}(z))]$ *as* $n \to \infty$.

Both limits are analytic functions of z *respectively on subsets* $D_{]-a_l,+\infty[} \setminus \{c \in [-a_l, \sup(-a_l, -a_u + 2\sqrt{q - a_u})], \, d = 0\} \bigcap B((0,0),K)$ *and* $D_{]-\infty,-a_u-2[} \setminus \{c \in [\inf(-a_u - 2, -a_l - 2\sqrt{q - a_l}), -a_u - 2], \, d = 0\} \bigcap B((0,0),K)$.

Strip of triangles with a double edge: G_n



The Tutte polynomial of G_n

$$T(G_n) = \frac{\mu_1 - 1}{\mu_1 - \mu_2}\mu_1^{n+1} + \frac{\mu_2 - 1}{\mu_2 - \mu_1}\mu_2^{n+1}.$$

On the hyperbola $(x-1)(y-1) = q$, these eigenvalues are of the form introduced before with $a = 1$.

$$T(G_n) = \alpha_1(z)[\lambda_1^+(z)]^{n+1} + \beta_1(z)[\lambda_1^-(z)]^{n+1}$$

with $\alpha_1(z) = \frac{\lambda_1^+(z)-1}{\lambda_1^+(z)-\lambda_1^-(z)}$ and $\beta_1(z) = \frac{\lambda_1^-(z)-1}{\lambda_1^-(z)-\lambda_1^+(z)}$.

Proposition

For the family of graphs $(G_n)_{n\geq 0}$, the location of the degeneration of the dominant eigenvalue is described in the z complex plane:

- $\forall\, q \geq 2$

$$|\lambda_1^+(z)| = |\lambda_1^-(z)| \Longleftrightarrow$$

$$d = 0,\ c \in [-1 - 2\sqrt{q-1}, -1 + 2\sqrt{q-1}]$$

- $\forall\, q \in [1, 2]$

$$|\lambda_1^+(z)| = |\lambda_1^-(z)| \Leftrightarrow \begin{cases} d = 0,\ c \in [-1 - 2\sqrt{q-1}, -1 + 2\sqrt{q-1}] \\ or\ z \in C((-q-1, 0), 2 - q) \end{cases}$$

Proposition

For the family of graphs $(G_n)_{n\geq 0}$, the location of the degeneration of the dominant eigenvalue is described in the v complex plane:

- *if* $q \in [1, 25/16]$

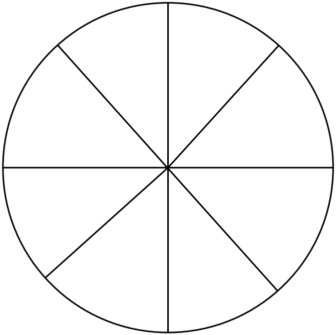
$$\begin{cases} \theta \in [\arccos(\frac{-1+2\sqrt{q-1}}{2\sqrt{q}}), \arccos(\frac{-1-2\sqrt{q-1}}{2\sqrt{q}})],\ r = 1 \\ \text{or } v \in F(C((-q-1,0), 2-q)). \end{cases}$$

- *if* $q \in [25/16, 2]$ $$\begin{cases} \theta \in [\arccos(\frac{-1+2\sqrt{q-1}}{2\sqrt{q}}), \pi],\ r = 1 \\ \text{or } \theta = \pi,\ r \in [1/r_1, r_1] \\ \text{or } v \in F(C((-q-1,0), 2-q)). \end{cases}$$

- *if* $q \geq 2$ $$\begin{cases} \theta \in [\arccos(\frac{-1+2\sqrt{q-1}}{2\sqrt{q}}), \pi],\ r = 1 \\ \text{or } \theta = \pi,\ r \in [1/r_1; r_1] \end{cases}$$

with r_1 and $1/r_1$ the roots of the polynomial
$\sqrt{q}r^2 - (1 + 2\sqrt{q-1})r + \sqrt{q} = 0.$

The wheel: B_n



The Tutte polynomial of B_n

$$T(B_n) = (xy - x - y - 1) + \mu_1^n + \mu_2^n.$$

On the hyperbola $(x-1)(y-1) = q$, these eigenvalues are of the form introduced before with $a = 1$ for $\{\mu_1, \mu_2\}$ and $a = q$ for the eigenvalue 1. The Tutte polynomial can be written as :

$$T(B_n) = (q-2)[\lambda_q^-(z)]^n + [\lambda_1^+(z)]^n + [\lambda_1^-(z)]^n$$

where $\lambda_q^-(z) = 1$.

Proposition

For the family of graphs $(B_n)_{n\geq 0}$, the location of the degeneration of the dominant eigenvalue is described in the z-complex plane:

- $\forall q \in [1,5]$, $q \neq 2$, when $|\lambda_1^+(z)| = |\lambda_1^-(z)| > 1$ or $|\lambda_1(z)| = 1$

$$\Rightarrow \begin{cases} c \in [-q, -1+2\sqrt{q-1}] & \text{and} \quad d = 0 \\ \text{or} & \\ c \in [-(q+5)/2, -q] & \text{and} \quad d^2 = -(c+q)^2 \frac{2c+q+5}{2c+q+1} \end{cases}$$

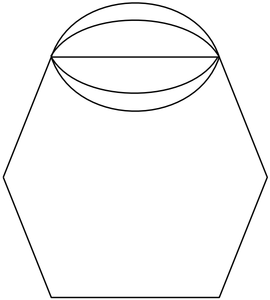
- $\forall q > 5$, when $|\lambda_1^+(z)| = |\lambda_1^-(z)| > 1$

$$\Rightarrow c \in [-1-2\sqrt{q-1}, -1+2\sqrt{q-1}] \quad \text{and}\, d = 0.$$

- For $q = 2$, when $|\lambda_1^+(z)| = |\lambda_1^-(z)|$

$$\Rightarrow c \in [-3, 1],\, d = 0.$$

Cycle with an edge of multiplicity n: C_n



The Tutte polynomial of C_n:

$$T(G_{n,n}) = \frac{xy - x - y}{(x-1)(y-1)}[x^n + y^n - 1] + \frac{(xy)^n}{(x-1)(y-1)}.$$

On the hyperbola $(x-1)(y-1) = q$, these eigenvalues are of the form introduced before with $a = 0$ for $\{x, y\}$ and with $a = q$ for $\{1, xy\}$.

$$T(G_{n,n}) = \frac{1}{q}[\lambda_q^+(z)]^n + \frac{1-q}{q}[\lambda_q^-(z)]^n + \frac{q-1}{q}[\lambda_0^+(z)]^n + \frac{q-1}{q}[\lambda_0^-(z)]^n.$$

Proposition

For the family of graphs $(G_{n,n})_{n\geq 0}$, the location of the degeneration of the dominant eigenvalue is described in the complex plane using respectively variable z and v as follows:
$\forall q \geq 1$, *when* $|\lambda_0(z)| = |\lambda_q(z)|$

$$\iff \left\{(c,d) \in \mathbb{R}^2,\ c \in [-2-q/2, -q/2],\right.$$
$$d^2 = -(2c+q+4)\left[\frac{(c+q)^2}{2c+q}\right]$$
$$\iff v \in \Delta_{-\sqrt{q}/2} \cup C((-1/\sqrt{q},0), 1/\sqrt{q})$$

where $\Delta_{-\sqrt{q}/2}$ *is the line* $Re(v) = -\sqrt{q}/2$ *and* $C((-1/\sqrt{q},0),1/\sqrt{q})$ *denotes the circle of center* $(-1/\sqrt{q},0)$ *and of radius* $1/\sqrt{q}$.

Concluding remarks

- Include different class of self dual graphs.

- Find other explicit form of eigenvalues.

- Conjecture for other infinite self dual graphs?

- Deep exploration of the antiferromagnetic region.