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Statistics for the contact process

Marta Fiocco*

Department of Mathematics and Computer Science, Leiden University, P.O. Box 9512, 2300 RA Leiden, The Netherlands

Willem R. van Zwet

EURANDOM

P.O. Box 513, 5600 MB Eindhoven, The Netherlands

A d-dimensional contact process is a simplified model for the spread of an infection on the lattice $\mathbb{Z}^d$. At any given time $t \geqslant 0$, certain sites $x \in \mathbb{Z}^d$ are infected while the remaining once are healthy. Infected sites recover at constant rate 1, while healthy sites are infected at a rate proportional to the number of infected neighboring sites. The model is parametrized by the proportionality constant λ. If λ is sufficiently small, infection dies out (subcritical process), whereas if λ is sufficiently large infection tends to be permanent (supercritical process).

In this paper we study the estimation problem for the parameter λ of the supercritical contact process starting with a single infected site at the origin. Based on an observation of this process at a single time t, we obtain an estimator for the parameter λ which is consistent and asymptotically normal as $t \to \infty$.

Key Words and Phases: contact process, supercritical contact process, statistical estimation.

1 The contact process: some properties

A d-dimensional contact process is a simplified model for the spread of a biological organism or an infection on the d-dimensional lattice $\mathbb{Z}^d$. At each time $t \geqslant 0$, every point of the lattice (or site) is either infected or healthy. As time passes, a healthy site is infected by each diseased site among its $2d$ immediate neighbors with Poisson rate λ; an infected site recovers and becomes healthy with Poisson rate 1. The processes involved are independent. If the process starts with a set $A \subset \mathbb{Z}^d$ of infected sites at time $t = 0$, then ξ_t^A will denote the set of infected sites at time $t \geqslant 0$ and $\{\xi_t^A : t \geqslant 0\}$ will denote the contact process. If the starting set is chosen at random according to a probability distribution α and independent of the further development of the process, then the process will be written as $\{\xi_t^\alpha : t \geqslant 0\}$.

The first thing to note about the contact process is that starting with a non empty set of infected sites at time $t = 0$, the infection will eventually die out with

* marta@math.leidenuniv.nl; vanzwet@math.leidenuniv.nl

probability 1 if $\lambda \leqslant \lambda_d$ for some critical value $\lambda_d \in (0, \infty)$. The infection will continue forever with positive probability if and only if $\lambda > \lambda_d$. Such a process is called *supercritical*. Thus, if we define the random hitting time

$$\tau^A = \inf\{t : \xi_t^A = \theta\}, \quad A \subset \mathbb{Z}^d, \tag{1}$$

with the convention that $\tau^A = \infty$ if $\xi_t^A \neq \theta$ for all $t \geqslant 0$, then for the supercritical contact process

$$\mathbb{P}(\tau^A = \infty) > 0 \tag{2}$$

for every non-empty $A \subset \mathbb{Z}^d$. Moreover, if A has infinite cardinality $|A|$, then

$$\mathbb{P}(\tau^A = \infty) = 1. \tag{3}$$

In the supercritical case, it is easy to show that the process $\xi_t^{\mathbb{Z}d}$ that starts with all sites infected, converges in distribution to the so-called upper invariant measure $v = v_\lambda$. Here convergence in distribution means convergence of probabilities of events defined by the behavior of the process on finite subsets of $\mathbb{Z}^d$, and 'invariant' refers to the fact that the process $\{\xi_t^v : t \geqslant 0\}$ is stationary. In particular, the distribution of ξ_t^v is equal to v for all t. Obviously, v is also invariant under integer valued translations of $\mathbb{Z}^d$. The long range behavior of the supercritical contact process $\{\xi_t^A : t \geqslant 0\}$ for arbitrary non-empty $A \subset \mathbb{Z}^d$ is described by the *complete convergence theorem* which asserts that for $\lambda > \lambda_d$ and $A \subset \mathbb{Z}^d$

$$\xi_t^A \xrightarrow{d} \mathbb{P}(\tau^A < \infty)\delta_\theta + \mathbb{P}(\tau^A = \infty)v_\lambda. \tag{4}$$

Here δ_θ assigns probability 1 to the empty set. Thus, given that the process survives forever, it converges in distribution to v_λ which depends on the dimension d and the value of λ, but not on the initial state A.

Next let us describe the so-called graphical representation of contact processes due to HARRIS (1978). This is a particular coupling of all contact processes of a given dimension d and with a given value of λ, but with every possible initial state A or initial distribution α. Consider space-time $\mathbb{Z}^d \times [0, \infty)$. For every site $x \in \mathbb{Z}^d$ we define on the line $x \times [0, \infty)$ a Poisson process with rate 1; for every ordered pair (x, y) of neighboring sites in $\mathbb{Z}^d$ we define a Poisson process with rate λ. All of these Poisson processes are independent.

We now draw a picture of $\mathbb{Z}^d \times [0, \infty)$ where for each site $x \in \mathbb{Z}^d$ we remove the points of the corresponding Poisson process with rate 1 from the line $x \times [0, \infty)$; for each ordered pair of neighboring sites (x, y) we draw an arrow going perpendicularly from the line $x \times [0, \infty)$ to the line $y \times [0, \infty)$ at the points of the Poisson process with rate λ corresponding to the pair (x, y). For any set $A \subset \mathbb{Z}^d$, define ξ_t^A to be the set of sites that can be reached by starting at time 0 at some site in A and traveling to time t along unbroken segments of lines $x \times [0, \infty)$ in the direction of increasing time, as well as along arrows. Clearly, $\{\xi_t^A : t \geqslant 0\}$ is distributed as a contact process with initial state A. By choosing the initial set at random with distribution α, we define $\{\xi_t^{\alpha} : t \geqslant 0\}$. The obvious beauty of this coupling is that for two initial sets of infected sites $A \subset B$, we have $\xi_t^A \subset \xi_t^B$ for all $t \geqslant 0$.

The contact process has the property of self-duality. If, in the graphical representation, time is run backwards and all arrows representing infection of one site by another, are reversed, then the new graphical representation has precisely the same probabilistic structure as the original one. In particular

$$\mathbb{P}(\xi_t^A \cap B = \theta) = P(\xi_t^B \cap A = \theta), \quad \text{for all } A, B \subset \mathbb{Z}^d \text{ and } t \geqslant 0. \tag{5}$$

By an abuse of notation we shall write ξ_t^A for the indicator function of ξ_t^A as well as for the set itself. Thus

$$\xi_t(x) = \begin{cases} 1 & \textit{if } x \textit{ is infected at time } t \\ 0 & \textit{if } x \textit{ is healthy at time } t. \end{cases} \tag{6}$$

With this notation (5) yields for $A = \{0\}$ and $B = \mathbb{Z}^d$,

$$\mathbb{P}(\tau^{\{0\}} \leqslant t) = \mathbb{P}(\xi_t^{\mathbb{Z}^d}(0) = 0)$$

and letting $t \to \infty$, this reduces to

$$\mathbb{P}(\tau^{\{0\}} < \infty) = \mathbb{P}(\xi_t^{\nu}(0) = 0)$$

because of the convergence in distribution of $\xi_t^{\mathbb{Z}^d}$ to ν. Combining these facts with an exponential bound on the speed of this convergence (see DURRETT 1991, page 5) we find

$$0 \leqslant \mathbb{P}(t < \tau^{\{0\}} < \infty) = \mathbb{P}(\xi_t^{\nu}(0) = 0) - \mathbb{P}(\xi_t^{\mathbb{Z}d}(0) = 0) \leqslant Ce^{-\gamma t} \tag{7}$$

For positive constants C and γ and all $t \geqslant 0$.

Another major result for the contact process is the shape theorem. *From this point on we shall assume that all contact processes are defined according to the graphical construction. We shall also restrict attention to the supercritical case*. In order to state the shape theorem we need some notation. Let $\|.\|$ denote the L^∞ norm on R^d, that is

$$\|x\| = \max_{1 \leqslant i \leqslant d} |x_i|$$

for $x = (x_1, \ldots, s_d) \in \mathbb{R}^d$, and let $Q = \{x \in \mathbb{R}^d : \|x\| \leqslant 1/2\}$ denote the unit hypercube centered at the origin. For $A, B \subset \mathbb{R}^d$, $A \oplus B = \{x + y : x \in A, y \in B\}$ will denote the direct sum of A and B and for real r, $rA = \{rx : x \in A\}$.

Define

$$Ht = \bigcup_{s \leqslant t} (\xi_s^{\{0\}}) \oplus Q, \tag{8}$$

$$K_t = \{x \in \mathbb{Z}^d : \xi_t^{\{0\}}(x) = \xi_t^{\mathbb{Z}d}(x)\} \oplus Q. \tag{9}$$

Thus for the process $\{\xi_t^{\{0\}} : t \geqslant 0\}$ that starts with a single infected site at the origin, H_t is obtained by taking the union of the sites that have been infected up to or at time t, and replacing these sites by unit hypercubes centered at these sites in order to fill up the space between neighboring sites. Similarly K_t is the filled-up version of the set of sites where $\xi_t^{\{0\}}$ and $\xi_t^{\mathbb{Z}d}$ coincide. We have (see BEZUIDENHOUT and, GRIMMETT 1990, DURRETT, 1991)

SHAPE THEOREM *There exists a bounded convex subset U of $\mathbb{R}^d$ with the origin as an interior point and such that for any $\varepsilon \in (0, 1)$,*

$$(1-\varepsilon)U \subset t^{-1}(H_t \cap K_t) \subset t^{-1}H_t \subset (1+\varepsilon)U, \tag{10}$$

eventually almost surely on the event $\{\tau^{\{0\}} = \infty\}$ where $\xi_t^{\{0\}}$ survives forever.

Roughly speaking the shape theorem asserts that, starting at the origin, the infection spreads linearly like tU and that shortly after the infection has arrived at a site, the distribution in the neighborhood of that site will approach the limiting distribution ν. The latter part of this statement is suggested by the fact that the two processes $\xi_t^{\{0\}}$ and $\xi_t^{\mathbb{Z}d}$ which approach the equilibrium process ξ_t^{ν} from below and from above, coincide on $(1-\varepsilon)tU$. As a result $\xi_t^{\{0\}}$ will be approximately distributed as ξ_t^{ν} on $(1-\varepsilon)tU$ for large t, provided it survives. Borrowing the vivid language of DURRETT and GRIFFEATH (1982), the infection $\xi_t^{\{0\}}$ spreads like a 'blob in equilibrium'. For a more precise and detailed account of the facts mentioned so far, the reader is referred to e.g. DURRETT and GRIFFEATH (1882), LIGGETT (1985, 1999), BEZUIDENHOUT and GRIMMETT (1990), DURRETT (1991) and also FIOCCO (1997). Figure 1 shows two simulations of the process $\xi_t^{\{0\}}$ for $\lambda = 0.5$ and $\lambda = 3$ after $N = 40{,}000$ transitions.

2 Statistical estimation

As was pointed out in Section 1, the contact process can serve as a simplified model for the spread of a biological organism, such as an infectious disease, or the growth of a forest. Obviously it is a rather simplified model, but it is clear that it may be refined to any reasonable degree by also admitting interactions between non-neighbors, infection and recovery rates varying in space or over time, more than two states at any site, etc. Such a class of models will have many applications in statistics and certainly deserves serious study. The contact process seems the obvious starting point for such studies: on the one hand it is the simplest non-trivial model in such a class and on the other, it exhibits the complex behavior of such processes in that it has more that one possible limit and produces a phase change at the critical value of λ. In short, a study of the statistical properties of the contact process should provide valuable information and insight for further studies of similar but more complex models.

The first statistical problem of interest for the contact process is to estimate the parameter λ on the basis of a realization of the process ξ_t at a single time t. In terms of the growth of a forest, one does not observe its entire history $\{\xi_t : t \geqslant 0\}$ but only what it looks like today. We also assume that the forest started with a single tree at the origin, so that we observe $\xi_t^{\{0\}}$. Obviously the exact distribution of the estimator

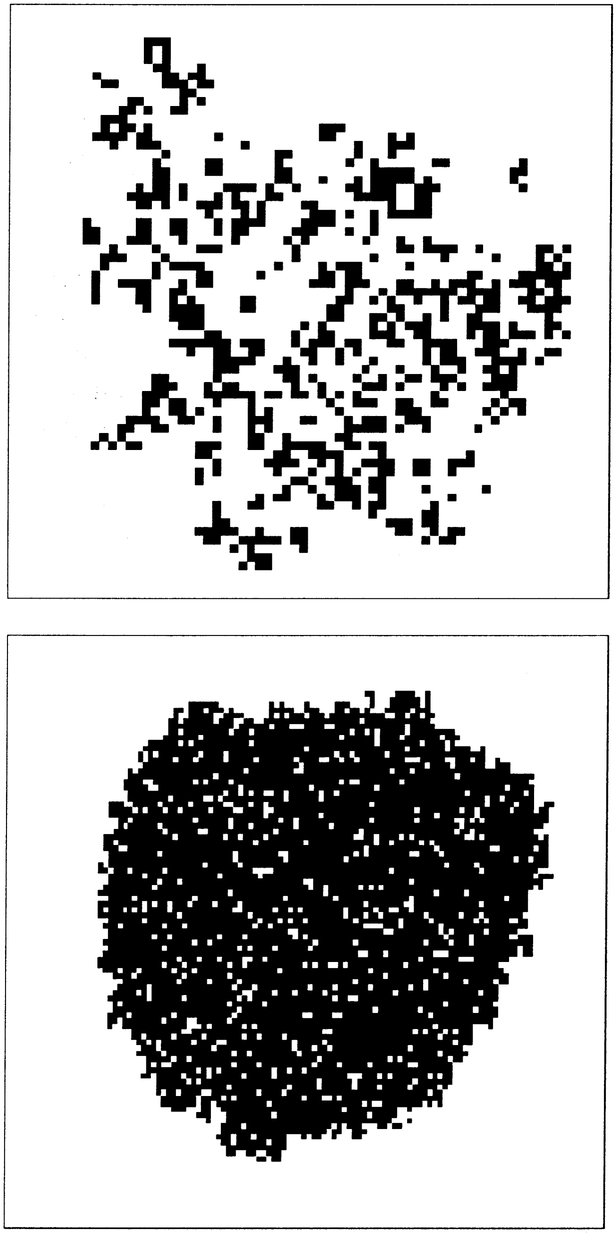

Fig. 1. The process $\xi^{\{0\}}_{t_N}$ for $\lambda = 0.5$ (upper) $\lambda = 3$ (lower) and $N = 40{,}000$.

$\hat{\lambda}_t$ for finite t will be intractable, but if possible, the estimator λ_t should be consistent and $t^{d/2}(\hat{\lambda}_t - \lambda)$ should have a manageable limit distribution as $t \to \infty$. We shall therefore restrict attention to large t. This seems realistic: if the forest deserves its name it has been there for quite some time.

As there is no standard method for obtaining estimators in a case like this, we shall simply use an estimating equation based on an equilibrium assumption for large t. At any site $x \in \mathbb{Z}^d$ and time $t \geqslant 0$, $\xi_t^{\{0\}}(x)$ decreases from 1 to 0 at rate $\xi_t^{\{0\}}(x)$ and increases from 0 to 1 at rate $\lambda(1 - \xi_t^{\{0\}}(x))\sum \xi_t^{\{0\}}(y)$, where the sum extends over the $2d$ neighbors of x in $\mathbb{Z}^d$. As we argued above, the shape theorem suggests that for large t the distribution of the process on the set $(1 - \varepsilon)tU$ should be close to its limit for $t \to \infty$. In the limit the rate of increase should equal the rate of decrease of the number of infected sites, so if $|z| = \sum |z_i|$ is the L^1 norm in $\mathbb{R}^d$, $D_t \subset (1 - \varepsilon)tU \cap \mathbb{Z}^d$ and

$$n_t^{\{0\}}(D_t) = \sum_{x \in D_t} \xi_t^{\{0\}}(x), \; k_t^{\{0\}}(D_t) = \sum_{x \in D_t} (1 - \xi_t^{\{0\}}(x)) \sum_{|x-y|=1} \xi_t^{\{0\}}(y), \tag{11}$$

then $n_t^{\{0\}}$ should be asymptotic to $\lambda k_t^{\{0\}}$ and hence $n_t^{\{0\}}/k_t^{\{0\}}$ should be a reasonable estimate of λ for large t. Note that $n_t^{\{0\}}(D_t)$ denotes the number of infected sites in D_t, whereas $k_t^{\{0\}}(D_t)$ is the number of pairs of infected and non-infected neighboring sites with the healthy site in D_t.

A problem in defining the estimator is the condition that the mask D_t should be a subset of $(1 - \varepsilon)tU \cap \mathbb{Z}^d$. The shape theorem only tells us that U is bounded and convex and contains the origin as an interior point. The set U also depends on the unknown λ. To make matters worse we may not know the value of t either: we know what time it is now, but when did the forest start? Luckily, the shape theorem ensures that for every $\varepsilon > 0$, $H_t \subset (1 + \varepsilon)tU$ eventually a.s. and if $\mathcal{C}(\xi_t^{\{0\}})$ denotes the convex hull of $\xi_t^{\{0\}}$, this implies that for every $\delta > 0$, there exists $\varepsilon > 0$ such that

$$C_t = (1 - \delta)\mathcal{C}(\xi_t^{\{0\}}) \cap \mathbb{Z}^d \subset (1 - \varepsilon)tU \cap \mathbb{Z}^d \qquad \textit{eventually almost surely.} \tag{12}$$

In view of this, we now define the estimator of λ at time t as

$$\hat{\lambda}_t^{\{0\}}(C_t) = \frac{n_t^{\{0\}}(C_t)}{k_t^{\{0\}}(C_t)} \tag{13}$$

Let $|A|$ denote the cardinality of a set $A \subset \mathbb{Z}^d$ and $N(0, \sigma^2)$ the normal distribution with expectation 0 and variance σ^2. We have the following result ensuring both consistency and asymptotic normality of the estimator $\hat{\lambda}_t^{\{0\}}(C_t)$, conditional on the process $\xi_t^{\{0\}}$ surviving forever.

THEOREM 1 *On the set* $\{\tau^{\{0\}} = \infty\}$ *where the process* $\xi_t^{\{0\}}$ *survives forever,* $\hat{\lambda}_t^{\{0\}}(C_t)$ *converges to* λ *in probability as* $t \to \infty$. *The conditional distribution of* $|C_t|^{1/2}(\hat{\lambda}_t^{\{0\}}(C_t) - \lambda)$ *given that* $\{\tau^{\{0\}} = \infty\}$, *converges weakly to* $N(0, \sigma^2)$ *as* $t \to \infty$. *Here* σ^2 *may be expressed in terms of the invariant measure* ν.

A number of remarks about the theorem are in order. The first concerns the condition $\{\tau^{\{0\}} = \infty\}$ which cannot be verified when we observe $\xi_t^{\{0\}}$ only at time t. However, in view of (7) we may replace the condition $\{\tau^{\{0\}} = \infty\}$ by $\{\tau^{\{0\}} > t\}$ which is easily verified at time t: just look whether there are any infected sites. Another problem may arise in the construction of the mask $C_t = (1 - \delta)C(\xi_t^{\{0\}}) \cap \mathbb{Z}^d$, which involves shrinking the convex hull of $\xi_t^{\{0\}}$ by a factor $(1 - \delta)$ towards the origin and selecting the lattice points of the resulting set. What if we don't know where the origin is located because we only see the forest, but not the first tree where it started? In this case we may either estimate the origin by some central point of the set $\xi_t^{\{0\}}$, or adopt a different type of shrinking, called peeling. This is done by taking the convex set $\mathcal{C}(\xi_t^{\{0\}})$ and peeling it like an apple by first removing all lattice points in the L^1-contour of the set, then constructing the convex hull of the remaining lattice points and repeating this procedure k times until a positive fraction α of the lattice points has been removed. The remaining lattice points now form C_t and the theorem will continue to hold as before.

Removing a positive fraction of the points of $\xi_t^{\{0\}}$ near the boundary of $\mathcal{C}(\xi_t^{\{0\}})$ before computing the estimator $n_t^{\{0\}}/k_t^{\{0\}}$ is an essential step in the estimation procedure. At these sites where the process has arrived only recently, the process is not yet in equilibirum. For each infected site the number of healthy neighbors is still too large which produces a negative bias in the estimator. Computer simulations confirm that the estimator $n_t^{\{0\}}(\mathbb{Z}^d)/k_t^{\{0\}}(\mathbb{Z}^d)$ based on all sites badly underestimates λ.

Removal of about 30% of the points of $\xi_t^{\{0\}}$ generally produces quite convincing estimates. Of course one faces the bias versus variance dilemma: the points one removes, the smaller the bias will become and the larger the variance. Also, fewer points need to be discarded for large values of λ and/or t. For small λ near the critical value the estimator becomes highly unstable. Mathematically speaking the situation is quite simple. The consistency statement of the theorem continues to hold even without shrinking, i.e. for $\delta = 0$ in (12), but in that case the asymptotic normality can no longer be proved with our methods and is probably no longer true.

It remains to discuss the unknown variance σ^2 in the theorem. There is an explicit expression σ^2 in terms of the invariant measure ν and one could determine $\sigma^2 = \sigma^2(\lambda)$ as a function of λ by simulation. In a particular case one would then use $\sigma^2(\hat{\lambda}_t^{\{0\}}(C_t))$ as an estimate of σ^2. However, it is also possible to estimate σ^2 from the observed $\xi_t^{\{0\}}$ itself by splitting the mask C_t into k approximately equal parts $C_{t,1}, \ldots, C_{t,k}$ and computing the values of $\hat{\lambda}_t^{\{0\}}(C_{t,i})$ for $i = 1, \ldots, k$. We then use $k^{-1}|C_t|$ times the sample variance of these values as an estimate of σ^2. The second method has the obvious advantage that it is not as dependent on the contact process model as the first. It is quite conceivable that $\hat{\lambda}_t^{\{0\}}(C_t)$ is a useful statistic in a much broader class of models than the contact process and in this case the second method is more likely to produce a sensible result.

Finally we remark that neither the estimator nor its asymptotic properties described in Theorem 1 change if $\xi_t^{\{0\}}$ is replaced by ξ_t^A for a finite A, i.e. it does not matter whether the forest starts with a single tree or with a few more.

3 Sketch of proofs

The proofs of the many steps that go into establishing the theorem are long, technical and involved. Here we only give a brief description of the main steps. Our theorem concerns the conditional distribution of the statistic $\hat{\lambda}_t^{\{0\}}(C_t)$ given $\{\tau^{\{0\}} = \infty\}$ as $t \to \infty$. The mask $C_t = (1-\delta)\mathcal{C}(\xi_t^{\{0\}}) \cap \mathbb{Z}^d$ is a random set which is eventually a.s. contained in $(1-\varepsilon)tU$ according to (12). However, the random set C_t is determined by the boundary of the convex hull $\mathcal{C}(\xi_t^{\{0\}})$, which is not contained in $(1-\varepsilon)tU$ for any $\varepsilon > 0$ eventualy a.s. according to the shape theorem. Thus $\hat{\lambda}_t^{\{0\}}(C_t)$ depends on what happens near the boundary of the blob of infected sites about which we know very little. This is the source of the main difficulty in the proof.

Let us therefore first prove the theorem with the random mask C_t replaced by a non-random mask $A_t \subset (1-\varepsilon)tU \cap \mathbb{Z}^d$ with $|A_t| \to \infty$. Thus we consider the conditional distribution of $\hat{\lambda}_t^{\{0\}}(A_t)$ given $\{\tau^{\{0\}} = \infty\}$. Now the statistic depends only on $\xi_t^{\{0\}} \cap (1-\varepsilon)tU$ and by the shape theorem we may replace $\xi_t^{\{0\}}$ by $\xi_t^{\mathbb{Z}^d}$ in the definition of our statistic. Of course the conditioning on the event $\{\tau^{\{0\}} = \infty\}$ which depends on the process $\{\xi_t^{\{0\}} : t \geqslant 0\}$ remains unchanged. However, by a relatively simple argument based on (7) one can show that we can replace conditonal probabilities for $\xi_t^{\mathbb{Z}^d}$ conditioned on $\{\tau^{\{0\}} = \infty\}$ by unconditional probabilities for $\xi_t^{\mathbb{Z}^d}$. Thus we have reduced a problem for the conditional distribution of $\xi_t^{\{0\}}$ given $\{\tau^{\{0\}} = \infty\}$ to the same problem for the unconditional distribution of $\xi_t^{\mathbb{Z}^d}$. If in analogy to (11) and (13) we define

$$n_t^{\mathbb{Z}^d}(A_t) = \sum_{x \in A_t} \xi_t^{\mathbb{Z}^d}(x),$$

$$k_t^{\mathbb{Z}^d}(A_t) = \sum_{x \in A_t} (1 - \xi_t^{\mathbb{Z}^d}(x)) \sum_{|x-y|=1} \xi_t^{\mathbb{Z}^d}(y), \; \hat{\lambda}_t^{\mathbb{Z}^d}(A_t) = \frac{n_t^{\mathbb{Z}^d}(A_t)}{k_t^{\mathbb{Z}^d}(A_t)}$$

then proving the theorem for non-random A_t reduces to proving the same results unconditionally for $\hat{\lambda}_t^{\mathbb{Z}^d}(A_t)$.

As the $\xi_t^{\mathbb{Z}^d}$ process is relatively easy to handle, we can show that for sets R_1 and R_2 in $\mathbb{Z}^d$ the correlation between functions of $\xi_t^{\mathbb{Z}^d} \cap R_1$ and $\xi_t^{\mathbb{Z}^d} \cap R_2$ decays exponentially with the distance of R_1 and R_2. This ensures a sufficient amount of independence between the terms of sums like $n_t^{\mathbb{Z}^d}(A_t)$ and $k_t^{\mathbb{Z}^d}(A_t)$ and together with bounds for their moments, it guarantees that the weak law and the central limit theorem will work for both sums after proper normalization as $\mathbb{E}n_t^{\mathbb{Z}^d}(A_t)/\mathbb{E}k_t^{\mathbb{Z}^d}(A_t) \to \lambda$ for $t \to \infty$ we obtain convergence in probability of $\hat{\lambda}_t^{\mathbb{Z}^d}(A_t)$ to λ, and λ also occurs in the normalization for the central limit theorem. This completes the proof for non random masks A_t.

Extending the proof to the random mask C_t is considerably harder. To make it work we have to show that, conditionally on $\{\tau^{\{0\}} = \infty\}$, C_t is asymptotically independent of $\xi_t^{\{0\}} \cap (1-\varepsilon)tU$. In other words we have to show that, given $\{\tau^{\{0\}} = \infty\}$, the choice of C_t which is determined by $\xi_t^{\{0\}}(x)$ for $x \notin (1-\varepsilon/2)tU$ is almost independent of $\xi_t^{\{0\}}(x)$ for $x \in (1-\varepsilon)tU$. As the two sets of sites are at a distance of order t from each other we now have to show that correlations for $\xi_t^{\{0\}}$ conditioned on $\{\tau^{\{0\}} = \infty\}$ decay sufficiently rapidly with increasing distance. It would take us too far to provide further details here. A complete proof of the theorem is given in FIOCCO and VAN ZWET (1998 and 1999).

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